Why is a linear diff. (Schrödinger) eq. associated to KdV?

- formalism to construct a linear diff eq associated to a non-linear one: Lax-method
- before introducing this: an intuitive answer for KdV:

KdV - mKdV relation:

– in order to construct the infinite tower of conserved quantities for the KdV we used a second non-lin. diff. eq., the mKdV:

$$\partial_t u - u\partial_x u - \partial_x^3 u = 0 \qquad \text{KdV}$$

$$+ \quad u(x,t) = -6\lambda + i\sqrt{6}\,\partial_x v(x,t) + v^2(x,t)$$

generalized Riccati trafo

$$\Rightarrow \quad \partial_t v - (-6\lambda + v^2)\partial_x v - \partial_x^3 v = 0 \qquad \text{mKdV}$$

(see p. 28 for $\lambda = 0$)

- from the mKdV + Galilei-trafo (ε) & Taylor expansion in ε we constructed all H_n's explicitly, so

$$\text{solution mKdV} \Rightarrow \text{solution KdV}$$

Q: When does "$\Leftarrow$" hold, i.e. when can one invert the Riccati trafo?

parametrise $\boxed{v(x,t) = i\sqrt{6}\,\frac{\partial_x \psi(x,t)}{\psi(x,t)}}$ (**) for $\psi(x,t)\neq 0$

$$\Rightarrow\ v^2 - i\sqrt{6}\,\partial_x v = -6\frac{(\partial_x\psi)^2}{\psi^2} - 6\left(\frac{\partial_x^2\psi}{\psi} - \frac{\partial_x\psi\,\partial_x\psi}{\psi^2}\right)$$

$$= -6\,\frac{\partial_x^2\psi}{\psi}$$

and thus

$$u(x,t) = -6\lambda - 6\,\frac{\partial_x^2\psi(x,t)}{\psi(x,t)}$$

$$\Longleftrightarrow\quad \boxed{\partial_x^2\psi(x,t) + \left(\tfrac{1}{6}u(x,t) + \lambda\right)\psi(x,t) = 0}$$

Schrödinger eq. with potent. $V(x) = -\frac{1}{6}u(x,t)$

- if $\psi(x,t)$ solves this Schrödinger eq. for given $u(x,t)$ we can reconstruct $v(x,t)$ via (**):

$$\partial_x\psi(x,t) = -\frac{i}{\sqrt{6}}\,v(x,t)\,\psi(x,t)$$

$$\Longleftrightarrow\quad \psi(x,t) = \exp\left[-\frac{i}{\sqrt{6}}\int^x dy\, v(y,t)\right] \qquad \square$$

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exercise: show that using this in the mKdV eq yields:

$$\psi^{-1}\left[\partial_t\psi + 4\lambda\,\partial_x\psi + \tfrac{1}{6}(\partial_x u)\psi - \tfrac{4}{3}u\,\partial_x\psi\right] = \text{const.}$$

- this is precisely eq. (*) on p. 37 which we used to determine the t-dependence of ψ via the inverse scattering method (Gel'fand-Levitan eq.), using $\square$ we then have $v(x,t)$ too

so $\quad \text{KdV} \xleftrightarrow[\text{eq.}]{\text{Schrödinger}} \text{mKdV}$

The Lax-Pair Method

Goal: given a nonlinear PDE in x and t:
→ find a linear operator L with constant, that is t-indep. eigenvalues, that relates both to the PDE and the Schrödinger eq.

- this method is not constructive, it just formalises (and facilitates) to find the Lax-pair of operators L, B
- to draw an analogy let's remind ourselves to

* time evolution of operators in quantum mechanics:

given time τ and a Hamilton operator $H = H^{+}$ that is not explicitly τ-dependent

⇒ time evolution is provided by the following unitary operator $U(\tau) = e^{-iH\tau}$, $U^{+}(\tau)U(\tau) = UU^{+} = \mathbb{1}$

- Schrödinger picture

* states τ-dependent:

$$i\partial_\tau \Psi(\tau) = H\,\Psi(\tau)$$

$$\Rightarrow \Psi(\tau) = U(\tau)\,\Psi(0)$$

* operators τ-indep:

- Heisenberg picture

* states τ-indep:

$$\Psi = \Psi(0)$$

* operators τ-dep:

$$A_H(\tau) = U^{+}(\tau)\,A_S\,U(\tau)$$

$$A_S\,\Psi(\tau) = \lambda\,\Psi(\tau) \quad \overset{U^{+}\cdot}{\Longleftrightarrow} \quad \boxed{A_H(\tau)\,\Psi(0) = \lambda\,\Psi(0)}$$

If we are seeking an operator with time indep. eigenvalues

$$\partial_\tau \lambda = 0 \Rightarrow \partial_\tau A_H(\tau) = 0 \quad , \quad U^\dagger(\tau) = e^{+iH\tau}$$

$$\Leftrightarrow 0 = \partial_\tau \left(U^\dagger(\tau) A_S U(\tau) \right) = \partial_\tau U^\dagger(\tau) A_S U(\tau) + U^\dagger \partial_\tau A_S U + U^\dagger A_S \partial_\tau U$$

$$= U^\dagger(\tau) \left(\partial_\tau A_S + iHA_S + A_S(-iH) \right) U(\tau)$$

$$\Leftrightarrow \quad \partial_\tau A_S = i[A_S, H] \qquad \left(\partial_\tau A_S = 0 \text{ by construction so } [A_S, H] = 0 \right)$$

def $B = -iH$ anti-Hermitian we have

$$\boxed{\partial_\tau A_S = [B, A_S]} \quad \text{I} \quad \text{"}\tau\text{-evolution of } A_S\text{"}$$

$$\partial_\tau \Psi = -iH\Psi \Rightarrow \quad \boxed{\partial_\tau \Psi(\tau) = B\Psi(\tau)} \quad \text{II} \quad \tau\text{-evolution of } \Psi$$

$$\boxed{A_S \Psi(\tau) = \lambda \Psi(\tau)} \quad \text{III} \quad \text{Schrödinger eq., eigenvalue eq.}$$

- the fact that λ is τ indep is equiv to the fact that $\exists$ unitary trafo of $A_H(\tau)$ that makes is τ-indep in that basis: $A_H \to U(\tau) A_H U^\dagger(\tau) = U U^\dagger A_S U U^\dagger = A_S$ τ indep

* Lax pair: $\tau \to t$ ($\neq$ Schrödinger time)

$$A_S \to L(t)$$

$$-iH \to B(t)$$

satisfying the same set of eqs.

Lax - Operator

* we are looking for an operator $L(t)$ that

  - is Hermitian $L(t)^\dagger = L(t) \Rightarrow$ real eigenvalues λ

  - has t-indep. eigenvalues

$$\boxed{L(t)\,\psi(t) = -\lambda\,\psi(t)} \quad \text{III}, \ \lambda \neq \lambda(t)$$

(otherwise we cannot determine the t-dependence of $\psi(t)$!)

$\Rightarrow \exists$ unitary operator $U(t)$: $\underline{U(t)^\dagger U(t) = U(t)\,U(t)^\dagger = \mathbb{1}}$

s.th. $\left\{\begin{array}{l} L(t) \to U(t)^\dagger L(t)\,U(t) \\ \psi(t) \to U(t)^\dagger \psi(t) \end{array}\right\}$ become t-indep.

$$\Rightarrow 0 = \partial_t (U^\dagger L U) = (\partial_t U^\dagger) L U + U^\dagger \partial_t L\, U + U^\dagger L\, \partial_t U$$

using $\underline{0 = \partial_t (U U^\dagger) = (\partial_t U) U^\dagger + U \partial_t U^\dagger}$

and defining $B(t) \equiv -U(t)\,\partial_t U^\dagger(t) = +(\partial_t U) U^\dagger$

we have $0 = U^\dagger\left(-U(\partial_t U^\dagger) L + \partial_t L + L(\partial_t U) U^\dagger\right) U$

$$\Leftrightarrow \quad \boxed{\partial_t L(t) = [B(t), L(t)]} \quad \text{I}$$

and $0 = \partial_t U + U(\partial_t U^\dagger) U$

$$\Leftrightarrow \quad \underline{\partial_t U(t) = +B(t)\,U(t)}$$

or $\partial_t (U(t)^\dagger \psi(t)) = 0 \Leftrightarrow \partial_t \psi(t) = -U(t)(\partial_t U^\dagger(t))\psi(t)$

$$\Leftrightarrow \quad \boxed{\partial_t \psi(t) = B(t)\,\psi(t)} \quad \text{II}$$

- the operators $L(t)$ and $B(t)$ are called <u>Lax pair</u> and the eq. I – III <u>Lax eq.</u>. Usually L and B are differential operators

* the art is to choose L, B s.th. <u>I is the non-linear PDE in $u(x,t)$</u> we want to solve (e.g. KdV) s.th. $L(t)$ is <u>linear</u> in $u(x,t)$

so that <u>III</u> defines a (linear) <u>Schrödinger eq</u> with potential $u(x,t)$ (B is not neccessarily linear in $u(x,t)$).

Then <u>eq. II</u> gives the <u>t-depencence</u> of the Schrödinger wave function $\psi(t)$ (as we did for KdV)

→ using inverse scattering & Gel'fand Levitan we can then find $u(x,t)$ from $u(x, t=0)$!

Examples for Lax pairs:

Choose $\boxed{L(t) = \partial_x^2 + \frac{1}{6} u(x,t)}$ III p. 56 is the Schrödinger eq we have solved for u being a solution of KdV

example 1)

with $\boxed{B_0(t) = \partial_x + \text{const}}$

$$\Rightarrow \quad \partial_t L = [B_0, L] = [\partial_x + \cancel{\text{const}}, \partial_x^2 + \frac{1}{6} u(x,t)]$$

$$\Leftrightarrow \quad \frac{1}{6}(\partial_t u) = \frac{1}{6}(\partial_x u) \quad \text{I}$$

chiral wave eq. (s. p. 34), linear PDE

with time evolution

$$\partial_t \Psi(x,t) = \partial_x \Psi(x,t) \quad \text{II}$$

example 2) choose the same $L(t)$ above

and ansatz $\boxed{B_1(t) = 4\,\partial_x^3 + \frac{1}{2}\left((\partial_x u) + 2u\partial_x\right) + \text{const}}$

these coeff. are chosen such that I will give KdV

* Note the similarity between the $B_{m=0,1}$ and the differential operators in the 2 Poisson brackets (e.g. p. 25)

* scaling behaviour: p. 31 $\dim[u] = -2 = \dim[\partial_x^2]$

$\Rightarrow$ $L(t)$ above has a definite scaling behaviour

$\Rightarrow$ $B_1(t)$ has scaling behaviour $\dim[\partial_x^3] = -3 = \dim[\partial_x u]$

it is the same as $[\partial_t] = -3$

In order to check that B_1 gives KdV from eq. I we need to compute the following commutators (exercise):

$$[\partial_x, u] = \partial_x u - u\partial_x = (\partial_x u) + u\partial_x - u\partial_x = (\partial_x u)$$

$$[\partial_x^2, u] = (\partial_x^2 u) + 2(\partial_x u)\partial_x$$

$$[\partial_x^3, u] = (\partial_x^3 u) + 3(\partial_x^2 u)\partial_x + 3(\partial_x u)\partial_x^2$$

$$[\partial_x^2, (\partial_x u)] = (\partial_x^3 u) + 2(\partial_x^2 u)\partial_x$$

$$[\partial_x^2, u\partial_x] = (\partial_x^2 u)\partial_x + 2(\partial_x u)\partial_x^2$$

$$[(\partial_x u), u] = 0$$

$$[u\partial_x, u] = u(\partial_x u)$$

and a constant commutes with everything

we also have $[\partial_x^k, \partial_x^l] = 0$ of course

We can now compute $\partial_t L = [B_1, L] \Leftrightarrow$ KdV:

- $6\partial_t L \cdot \mathbb{1} = 6\partial_t(\partial_x^2 + \frac{1}{6}u(x,t))\mathbb{1} = \boxed{(\partial_t u(x,t))}$
- $6[B_1, L]\mathbb{1} = 6[4\partial_x^3 + \frac{1}{2}(\partial_x u) + u\partial_x + \cancel{\text{const}}, \partial_x^2 + \frac{1}{6}u]\mathbb{1}$

$$= 4\{(\partial_x^3 u) + 3(\partial_x^2 u)\partial_x + 3(\partial_x u)\partial_x^2\}\mathbb{1}$$

$$+ 3\{-(\partial_x^3 u) - 2(\partial_x^2 u)\partial_x\}\mathbb{1} - 6\{(\partial_x^2 u)\partial_x + 2(\partial_x u)\partial_x^2\}\mathbb{1}$$

$$+ u(\partial_x u)\mathbb{1} = \boxed{\partial_x^3 u + u\partial_x u}$$

I $\Leftrightarrow$ KdV ✓

$$\text{II}: \quad \partial_t \psi = B_1 \psi(t)$$
$$= 4\partial_x^3 \psi + \frac{1}{2}(\partial_x u)\psi + u\partial_x \psi + \text{const}\,\psi$$

using ∂_x III (Schrödi): $\quad \partial_x^3 \psi + \frac{1}{6}(\partial_x u)\psi + \frac{1}{6} u \partial_x \psi = -\lambda \partial_x \psi$

we obtain

$$\partial_t \psi = -4\lambda \partial_x \psi - \frac{1}{6}(\partial_x u)\psi + \frac{1}{3} u \partial_x \psi + \text{const}\,\psi$$

for the time evolution of $\psi(x,t)$, which equals eq. (*) on p 37.

Lax-Pairs of the KdV Hierarchy

recall (pages 33+34) the chain of eqs:

$$\partial_t u = \{u(x), H_n\}_1 = \{u(x), H_{n+1}\}_2$$

$$\Leftrightarrow \quad \boxed{\partial_t u = \partial_x \frac{\delta H_{n+1}}{\delta u(x)}}$$

$n=0 \qquad \partial_t u = \partial_x u \qquad$ chiral wave eq.

$n=1 \qquad \partial_t u = \partial_x^3 u + u\partial_x u \qquad$ KdV

$n=2 \qquad \partial_t u = \partial_x^5 u + \frac{4}{3} u \partial_x^3 u + \frac{10}{3}(\partial_x u)(\partial_x u^2) + \frac{5}{6} u^2 \partial_x u$

etc.

- One can show that for each of these non-lin PDE of the KdV hierarchy one can construct an operator $B_n(t)$ s.th. $\boxed{\partial_t L = [B_n, L]}$ gives the n-th eq.

Remarks: * all these non-linear PDE can be solved with the same Schrödinger eq.

$$\text{III} \quad L\,\psi(t) = -\lambda\,\psi(t)$$

with our $L(t) = \partial_x^2 + \frac{1}{6}\,u(x,t)$

using the inverse scattering method.

* However, for each n the t-evolution of $\psi(x,t)$ changes and is given by

$$\text{II} \quad \partial_t \psi(x,t) = B_n(t)\,\psi(x,t)$$

Exercise: determine $B_{n=2}$

Note that because $\partial_t L = \frac{1}{6}\partial_t u = \partial_x \frac{\delta H_{n+1}}{\delta u(x)}$

$$= [B_n, L]$$

and the fact that H_{n+1} has a defined scaling behaviour (p. 31: $\dim[H_n] = -2(n+1)-1$) also B_n has a defined scaling behaviour (which?).

This allows to determine the coefficient functions $b_j(u)$ in the Ansatz

$$B_n(t) \sim \partial_x^{2n+1} + \sum_{j=1}^{n}\left(b_j(u)\,\partial_x^{2j-1} + \partial_x^{2j-1}\,b_j(u)\right)$$