Lax Pair & First Order Formulation

- general theory of differential equations:

  a linear diff. eq. of n-th order

  $\Leftrightarrow$ linear system of n eqs. of 1st order

* here: map Schrödinger eq. = 2nd order

$$\partial_x^2 \psi + V\psi + \lambda\psi = 0$$

to a 2×2 matrix diff. eq. of 1st order

$\Leftrightarrow$ seek matrix-Lax-operators with

$$L \sim (\quad)\cdot\partial_x \qquad \text{rather than } \sim\partial_x^2$$

motivation:

* this will give a simpler generalisation of classical integrable systems to quantum mech. integrable systems in this formalism

* in particular we will encounter Lax pairs for the following nonlinear PDE's (apart from KdV):

  - non linear Schrödinger eq.
  - sine and sinh-Gordon eq

· Example: Schrödinger eq. as a warmup:

2nd order $\underline{\partial_x^2 \psi + V\psi = -\lambda \psi}$ define $\psi_1 \equiv \psi$, $\psi_2 \equiv \partial_x \psi$

$$\Longleftrightarrow \left\{ \begin{array}{l} \partial_x \psi_1 = \psi_2 \\ \partial_x \psi_2 + V\psi_1 = -\lambda \psi_1 \end{array} \right\}$$

$$\Longleftrightarrow \quad \begin{array}{l} \partial_x \psi_1 - \psi_2 = 0 \\ \partial_x \psi_2 + (V+\lambda)\psi_1 = 0 \end{array} \quad \Longleftrightarrow \quad \underline{\begin{pmatrix} \partial_x & -1 \\ V(x)+\lambda & \partial_x \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0}$$

1st order formulation

* note that this is not of the form $L \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \lambda \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \rightarrow$ modify $\overline{\text{III}}$ to $L \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$ $\tilde{\underline{\text{III}}}$

⇝ Make a General Ansatz for L and B of size 2×2:

- use $\mathbb{1}_{2\times 2}$ and Pauli matrices $\sigma_{\ell=1,2,3}$ as a basis for all Hermitian 2×2 matrices

$$\mathbb{1}_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the σ_j are Hermitian and traceless and enjoy the commutation relations $\boxed{[\sigma_k, \sigma_\ell] = 2i\,\varepsilon_{k\ell m}\,\sigma_m}$ $k,\ell,m \in \{1,2,3\}$

with $\varepsilon_{123} = 1$, antisym. in all indices

Note: the Pauli matrices form the algebra $\underline{su(2)}$

that generates the group $SU(2)$ of

special (S) : $\det U = 1$

unitary (U) : $UU^\dagger = U^\dagger U = 1$

2x2 matrices (2) : $U = e^{i\alpha_j\sigma_j}$

define $\sigma_\pm \equiv \frac{1}{2}(\sigma_1 \pm i\sigma_2) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{"+"} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \text{"−"} \end{cases}$

* note that the $S_k = \frac{1}{2}\sigma_k$ are the components of spin $\frac{1}{2}$ op., with $S_\pm$ being raising and lowering operators

- properties of σ_k:
  - $\sigma_k^2 = \mathbb{1} \quad \forall k = 1, 2, 3$ (no summation)

$$\Rightarrow \qquad \sigma_+^2 = \frac{1}{4}(\sigma_1 + i\sigma_2)(\sigma_1 + i\sigma_2)$$

$$= \frac{1}{4}(\sigma_1^2 + i\sigma_1\sigma_2 + i\sigma_2\sigma_1 - \sigma_2^2)$$

$$= \frac{1}{4}(\mathbb{1} + i(i\sigma_3) + i(-i\sigma_3) - \mathbb{1}) = 0$$

as also $\sigma_1\sigma_2 = i\sigma_3$ and cyclic permutations (reverse)

- ditto $\sigma_-^2 = 0$
- $\sigma_\pm \sigma_3 \underset{\text{comm. rel}}{=} -\sigma_3\sigma_\pm = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sigma_\pm = \begin{cases} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = -\sigma_+ & \text{"+"} \\ \begin{pmatrix} 0 & 0 \\ +1 & 0 \end{pmatrix} = +\sigma_- & \text{"−"} \end{cases}$
- $\sigma_\pm \sigma_\mp = \frac{1}{2}(\mathbb{1} \pm \sigma_3)$

General ansatz for the Lax operator $\mathcal{L}$:

$$\tilde{\mathrm{I}} \qquad \boxed{\mathcal{L}\,\phi \equiv \left(\mathbb{1}\,\partial_x - q(x,t)\,\sigma_+ - r(x,t)\,\sigma_- + i\zeta\,\sigma_3 \right)\phi = 0}$$

spectral parameter

where $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

$$\Leftrightarrow \quad \partial_x \phi = \left(q\sigma_+ + r\sigma_- - i\zeta\sigma_3 \right)\phi$$

$$\tilde{\mathrm{II}} \qquad \boxed{\partial_t \phi = \mathcal{B}\,\phi \equiv \left(P(x,t)\,\sigma_+ + Q(x,t)\,\sigma_- + R(x,t)\,\sigma_3 \right)\phi}$$

* the functions P, Q, R and r, q, ζ are not indep as it holds

$$\boxed{\partial_t \mathcal{L} = [\mathcal{B}, \mathcal{L}]}$$

lhs: $\partial_t \mathcal{L}\cdot\mathbb{1} = -\partial_t q\,\sigma_+ - \partial_t r\,\sigma_-$ for ζ t-indep

rhs: use $[\sigma_3, \sigma_\pm] = \pm 2\sigma_\pm$ and $[\sigma_+, \sigma_-] = \sigma_3$ (ex.)

$$\begin{aligned}
[\mathcal{B}, \mathcal{L}]\mathbb{1} &= [\, P\sigma_+ + Q\sigma_- + R\sigma_3 \,,\, \partial_x - q\sigma_+ - r\sigma_- + i\zeta\sigma_3 \,]\mathbb{1} \\
&= -\partial_x P\sigma_+ - \partial_x Q\sigma_- - \partial_x R\sigma_3 \qquad + 2iQ\zeta\sigma_- \\
&\quad + Qq\,\sigma_3 - 2Rq\,\sigma_+ - Pr\,\sigma_3 + 2Rr\sigma_- - 2iP\zeta\sigma_+
\end{aligned}$$

because $\sigma_+, \sigma_-, \sigma_3$ also form a basis we can compare coeffs

$$\boxed{\begin{aligned}
-\partial_t q &= -\partial_x P - 2Rq - 2iP\zeta && a) \\
-\partial_t r &= -\partial_x Q + 2Rr + 2iQ\zeta && b) \\
0 &= -\partial_x R + Qq - Pr && c)
\end{aligned}}$$

(*)

* Note: the same set of eqs also follows from the compatibility condition

$$\boxed{\partial_t \partial_x \phi = \partial_x \partial_t \phi} \qquad \text{(exercise)}$$

$$\Leftrightarrow \partial_t (q\,\sigma_+ + r\sigma_- - i\varsigma\sigma_3)\phi = \partial_x (P\sigma_+ + Q\sigma_- + R\sigma_3)\phi$$

our set of Matrix Lax eqs is with L a 1st order diff. op.

$$\boxed{\begin{aligned} \partial_t L &= [B, L] \\ \partial_t \phi &= B\phi \\ L\phi &= 0 \end{aligned}}$$

important: for any set of functions P, Q, R and q, r that satisfy (*) we have a matrix Lax pair L, B, and a corresponding integrable non-linear PDE!

example KdV:

choose $r(x,t) = \text{const} = 6, \quad q(x,t) = -\frac{1}{36} u(x,t)$

$\Rightarrow$ (*) yields: b) $R(x,t) = \frac{1}{12}\partial_x Q(x,t) - \frac{i\varsigma}{6} Q(x,t)$

c) $P(x,t) = \frac{1}{6} Q q - \frac{1}{6}\partial_x R$

$$= \frac{1}{6} Q q - \frac{1}{72}\partial_x^2 Q + \frac{i\varsigma}{36}\partial_x Q$$

a) $\partial_t q = \partial_x P + 2qR + 2i\varsigma P$

$$= -\frac{1}{72}\partial_x^3 Q + \frac{1}{6}(\partial_x Q) q + \frac{1}{6} Q(\partial_x q) + \frac{i\varsigma}{36}\partial_x^2 Q$$

$$+ \frac{q}{6}\partial_x Q - q\frac{i\varsigma}{3} Q + \frac{i}{3}\varsigma Q q - \frac{i\varsigma}{36}\partial_x^2 Q - \frac{\varsigma^2}{18}\partial_x Q$$

$$\Leftrightarrow \quad \boxed{\partial_t u = \frac{1}{2}\left(\partial_x^3 + \frac{1}{3}(\partial_x u + u\partial_x)\right)Q + 2\zeta^2 \partial_x Q} \quad (1)$$

- choose furthermore $Q(x,t) = 2u(x,t)$, $\zeta \equiv 0$:

$\Rightarrow$ (1) gives the KdV eq:

$$\partial_t u = \partial_x u^3 + \frac{1}{3}\partial_x u^2 + u\partial_x u = \partial_x^3 u + u\partial_x u$$

with Lax op

$$\boxed{\begin{aligned} L &= \partial_x + \frac{1}{36}u(x,t)\sigma_+ - 6\sigma_- \\ B &= \left(\frac{u^2}{108} - \frac{1}{36}(\partial_x^2 u)\right)\sigma_+ + 2u\sigma_- + \frac{1}{6}\partial_x u\,\sigma_3 \end{aligned}}$$

and $\partial_t L = [B, L] \Leftrightarrow$ KdV (check it!)

- the mKdV eq can be obtained in a similar way

<u>Generation of KdV hierarchy from (1):</u>

choose $Q(x,t) = 2\sum_{j=0}^{n} Q_j(u)\left(-4\zeta^2\right)^{n-j}$ instead

expand (1) in powers of ζ^2:

$$\Rightarrow \quad \boxed{\begin{aligned} &\left(\partial_x^3 + \frac{1}{3}(\partial_x u + u\partial_x)\right)Q_j = \partial_x Q_{j+1} \quad \text{for } j = 0, 1, \dots, n-1 \\ &\qquad \text{setting } Q_0 \equiv 1 \\ &\text{and } \partial_t u = \left(\partial_x^3 + \frac{1}{3}(\partial_x u + u\partial_x)\right)Q_n \end{aligned}}$$

which implies $Q_j(u) \sim \frac{\delta H_j}{\delta u(x)}$

* we already saw in the 2n order formalism ($L \sim \partial_x^2$) that with L and B_n we can generate the entire KdV hierarchy!

Lax pairs of other nonlinear PDE's

cubic interact.

* nonlinear Schrödinger eq $\quad i\partial_t\psi = -\partial_x^2\psi + 2\kappa\,|\psi|^2\psi$

$\kappa\in\mathbb{R}$ const.

back to ansatz on page 65:

choose $\boxed{\begin{aligned} q(x,t) &= \sqrt{\kappa}\,\psi^*(x,t)\\ r(x,t) &= \sqrt{\kappa}\,\psi(x,t)\end{aligned}}$; $\quad \kappa>0 \Rightarrow \sqrt{\kappa}\in\mathbb{R}$
$\quad \kappa<0 \Rightarrow \sqrt{\kappa} = i\sqrt{|\kappa|}\in i\mathbb{R}$

$$\Rightarrow \quad q^*(x,t) = \begin{cases} (i\sqrt{|\kappa|}\psi^*)^* & \kappa<0\\ (\sqrt{\kappa}\psi^*)^* & \kappa>0\end{cases} = \begin{cases} -i\sqrt{|\kappa|}\,\psi & \kappa<0\\ \sqrt{\kappa}\,\psi & \kappa>0\end{cases}$$

$$= \operatorname{sign}(\kappa)\cdot r(x,t)$$

a)* $\Rightarrow \quad \partial_t q^* = \partial_x P^* + 2R^* q^* - 2iP^*\zeta$

vs b) : $\quad \partial_t r = \partial_x Q - 2Rr - 2iQ\zeta$

consistency requires $\boxed{Q = \operatorname{sign}(\kappa)\,P^*}$ and $\boxed{R^* = -R}$

choose $\boxed{R = 2i\zeta^2 + i\kappa\,\psi^*\psi}$

$\xRightarrow{c)}$ $\quad 0 = -i\kappa\,\partial_x(\psi^*\psi) + Q\sqrt{\kappa}\,\psi^* - \operatorname{sign}(\kappa)\,Q^*\sqrt{\kappa}\,\psi$

ζ const

Solution : $\boxed{Q = i\sqrt{\kappa}\,\partial_x\psi + z\sqrt{\kappa}\,\psi}$

$\Rightarrow Q^* = -i\operatorname{sign}(\kappa)\sqrt{\kappa}\,\partial_x\psi^* + z\operatorname{sign}(\kappa)\sqrt{\kappa}\,\psi^*$

where $z = \text{const}\in\mathbb{R}$ arbitrary. $\quad$ choose $z = -2\zeta$

$$\Rightarrow \text{b)} \quad -\cancel{\sqrt{\kappa}}\,\partial_t\psi = -\partial_x\left(i\cancel{\sqrt{\kappa}}\,\partial_x\psi + \zeta\cancel{\sqrt{\kappa}}\,\psi\right)$$
$$+\,2\left(2i\zeta^2 + i\kappa\,\psi^*\psi\right)\cancel{\sqrt{\kappa}}\,\psi$$
$$+\,2i\left(i\cancel{\sqrt{\kappa}}\,\partial_x\psi + \zeta\cancel{\sqrt{\kappa}}\,\psi\right)\zeta$$

$$\Leftrightarrow \quad \underline{-\partial_t\psi = -i\partial_x^2\psi + 2i\kappa|\psi|^2\psi}$$

$\Rightarrow$ $\cdot(-i)$ non-linear Schrödinger eq. above, indep. on choice of $\underline{\underline{\zeta}}$.