$\Rightarrow$ p65 b): $\quad \partial_t r = \partial_x Q - 2Rr - 2i\xi Q$

$$\Leftrightarrow \quad \frac{1}{2}\partial_t\partial_x\psi = \frac{i}{4\xi}\partial_x \sin(h)[\psi] - \cancel{4}\frac{i}{4\xi}\cos(h)[\psi]\frac{1}{\cancel{4}}\partial_x\psi$$
$$- (2i\xi)\frac{i}{4\xi}\sin(h)[\psi]$$

$$\Leftrightarrow \quad \begin{cases} \partial_t\partial_x\psi = \sin[\psi] \\ \partial_t\partial_x\psi = \sinh[\psi] \end{cases}$$

Sine - Gordon

sinh - Gordon

in light cone coordinates (see below)

ex: get same from a)

remark

* it is clear, that having a Lax Pair for one of the signs we also have a pair for the other sign by a simple rotation $\psi \Leftrightarrow i\psi$ as

- $\sinh[\psi] = \frac{1}{2}(e^{+\psi} - e^{-\psi})$

  with $r = -q$, $Q = +P \sim i\sin$, $R \sim i\cos$

- $\sin[\psi] = \frac{1}{2i}(e^{i\psi} - e^{-i\psi})$

  $\psi \to i\psi$ & multiply eqs (*) by i

  $\to r = +q$, $Q = -P \sim i\sinh$, $R \sim i\cosh$

Origin of the sine-Gordon eq.

* rotate back from light cone coordinates to ordinary Minkowski space: $\partial_x \partial_t = -\partial_\tau^2 + \partial_y^2$

* let $\Psi = \Psi(y, \tau)$ be a function of time τ and space y in Minkowski space

def
$$\left.\begin{aligned} x &\equiv \tfrac{1}{2}(y+\tau) \\ t &\equiv \tfrac{1}{2}(y-\tau) \end{aligned}\right\}$$
light-cone coord.

* write $\Psi(y = x+t,\ \tau = x-t) \equiv \Psi(x,t)$

correct
$$\Rightarrow \quad \partial_\tau \Psi(x,t) = \tfrac{1}{2}\partial_x \Psi - \tfrac{1}{2}\partial_t \Psi = \tfrac{1}{2}(\partial_x - \partial_t)\Psi$$
$$\partial_y \Psi(x,t) = \tfrac{1}{2}\partial_x \Psi + \tfrac{1}{2}\partial_t \Psi = \tfrac{1}{2}(\partial_x + \partial_t)\Psi$$

$$\Rightarrow \underbrace{\left(\partial_\tau^2 - \partial_y^2\right)}_{\square \text{ in 2D}} \Psi(x,t) = \tfrac{1}{4}\left(\partial_x^2 - \partial_t\partial_x - \partial_x\partial_t + \partial_t^2 - \partial_x^2 - \partial_x\partial_t - \partial_t\partial_x - \partial_t^2\right)\Psi$$
$$= -\partial_x\partial_t\Psi = -\partial_t\partial_x\Psi$$

assuming Ψ has continuous derivatives $\Rightarrow [\partial_x, \partial_t] = 0$

$$\Rightarrow \quad \partial_x\partial_t\Psi - \sin[\Psi] = 0 \iff \boxed{\left(\partial_\tau^2 - \partial_y^2\right)\Psi + \sin[\Psi] = 0}$$

sine-Gordon eq in 2D Minkowski

linearise $|\Psi| \ll 1$: $\quad (\partial_\tau^2 - \partial_y^2)\Psi + \Psi = 0$

Klein-Gordon with all parameters incl mass $= 1$

$$\rightarrow \left(\tfrac{1}{c^2}\partial_\tau^2 - \nabla^2\right)\Psi + \frac{m^2c^2}{\hbar^2}\Psi = 0$$ relativistic wave eq.

($\neq$ Schrödinger eq: non rel)

Non linear Schrödinger equation as a Hamiltonian System

- in the following we will investigate the NLS eq. in more detail by:
  - showing that is has 2 Poisson brackets
  - investigating its time evolution via inverse scattering

Hamiltonian $\qquad H[\psi] = \int_{-\infty}^{\infty} dx \left\{ (\partial_x \psi^*)(\partial_x \psi) + \kappa (\psi^* \psi)^2 \right\}$

with Poisson bracket $\{\,,\}_2$ defined as

$$\{\psi(x,t), \psi^*(y,t)\}_2 = -i\,\delta(x-y) \qquad (\psi \in \mathbb{C})$$

$$\{\psi(x,t), \psi(y,t)\}_2 = 0 = \{\psi^*(x,t), \psi^*(y,t)\}_2$$

$\Rightarrow$ ψ, ψ^* are conjugate variables

- antisymmetry: $\{\psi^*(y,t), \psi(x,t)\}_2 = \left(\{\psi(x,t), \psi^*(y,t)\}_2\right)^* = +i\,\delta(x-y)$

- for general operators we have

$$\{A, B\}_2 = \int dx\,dy \left(\frac{\delta A}{\delta \psi(x)} (-i)\delta(x-y) \frac{\delta B}{\delta \psi^*(y)} - (A \leftrightarrow B) \right)$$

Equations of motion $\Leftrightarrow$ NLS:

$$\partial_t \psi(z,t) = \{\psi(z,t), H\}_2 = \int dx\,dy\, \underbrace{\frac{\delta \psi(z,t)}{\delta \psi(x,t)}}_{=\delta(z-x)} (-i)\delta(x-y) \frac{\delta H}{\delta \psi^*(y,t)} - \underbrace{\frac{\delta \psi(z,t)}{\delta \psi^*(y,t)}}_{=0} (-i)\delta(x-y) \frac{\delta H}{\delta \psi(x,t)}$$

$$\Leftrightarrow \boxed{\partial_t \psi(z,t) = -i\left(-\partial_z^2 \psi + 2\kappa (\psi^* \psi)\psi\right)}$$

NLS eq on p. 69

Conserved quantities and 2. Poisson bracket

- $\underline{H_1 = \int\limits_{-\infty}^{\infty} dx\, \psi^* \psi}$ particle number

as $\partial_t H_1 = \{H_1, H\}_2 = \int dx dy \left[\psi^*(x) (-i) \delta(x-y) \left(-\partial_y^2 \psi + 2\kappa \psi^* \psi \psi(y) \right) \right.$
$\left. - \psi(x) (-i) \delta(x-y) \left(-\partial_x^2 \psi^* + 2\kappa \psi^* \psi \psi^*(x) \right) \right]$
$= 0$ after integration by parts

- $\underline{H_2 = \int\limits_{-\infty}^{\infty} dx \left(\psi^* \partial_x \psi - (\partial_x \psi^*) \psi \right)}$ momentum

exercise: check $\partial_t H_2 = 0$

- H itself is conserved (as not explicitly t-dep)

$\partial_t H = \{H, H\}_2 = 0$ energy

- second Poisson bracket $\{\,,\,\}_1$:

$$\{\psi(x,t), \psi^*(y,t)\}_1 \equiv (-i) \left(\partial_x^2 + 2\kappa \psi^*(x,t) \psi(x,t) \right) \delta(x-y)$$

and $\{\psi^*(x,t), \psi^*(y,t)\}_1 = 0 = \{\psi(x,t), \psi(y,t)\}_1$

$$\Rightarrow \underline{\partial_t \psi} = \{\psi(z,t), H_1\}_1 = \int dx dy\, \delta(z-x) (-i) \left(\partial_x^2 + 2\kappa \psi^*(x,t) \psi(x,t) \right) \delta(x-y) \cdot \psi(y,t)$$

$$\underline{= -i \left(\partial_z^2 \psi(z,t) + 2\kappa \psi^*(z,t) \psi(z,t)^2 \right)} \quad \Leftrightarrow \text{NLS}$$

$\rightarrow$ from these 2 Poisson bracket we could construct again an ∞-tower of conserved quantities

Time development of the scattering data: 1st order formalism

- we will study the asymptotic t-dependence of ϕ in the 1st order Formalism, using the example of the NLS eq.:

* we have to distinguish signs of κ: for $\kappa<0$ there will be scattering states possible

recall (p 68):

$$q = i\sqrt{|\kappa|}\,\Psi^* \,, \quad r = i\sqrt{|\kappa|}\,\Psi \qquad\qquad 0$$

$$P = \sqrt{|\kappa|}\,\partial_x \Psi^* - 2i\varsigma\sqrt{|\kappa|}\,\Psi^* \qquad\qquad 0$$

$$Q = -\sqrt{|\kappa|}\,\partial_x \Psi - 2i\varsigma\sqrt{|\kappa|}\,\Psi \quad \xrightarrow{|x|\to\infty} \quad 0$$

$$R = 2i\varsigma^2 - i|\kappa|\Psi^*\Psi \qquad\qquad 2i\varsigma^2$$

* requiring that Ψ, Ψ^* and their derivatives vanish asymptotically we have

$\Rightarrow$ the Lax eqs read asymptotically:

- $\partial_t \phi = B\phi = (P\sigma_+ + Q\sigma_- + R\sigma_3 + \overset{\text{const}}{C})\,\phi$

$$\rightarrow \boxed{\partial_t \phi = (2i\varsigma\sigma_3 + C)\,\phi} \quad , \text{ with } C \text{ determined by boundary cond}$$

- before we analyse

$$0 = L\phi = (\partial_x - q\sigma_+ - r\sigma_- + i\varsigma\sigma_3)\,\phi$$

asymptotically we exploit some of its symmetries:

here for NLS: $\left(\partial_x - i\sqrt{|\kappa|}\,(\psi^* \sigma_+ + \psi \sigma_-) + i\zeta \sigma_3\right)\phi = 0$ ①

$$\Longleftrightarrow \left(\partial_x + i\sqrt{|\kappa|}\begin{pmatrix} 0 & \psi^* \\ \psi & 0 \end{pmatrix} + i\begin{pmatrix} \zeta & 0 \\ 0 & -\zeta \end{pmatrix}\right)\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0$$

$$\Longleftrightarrow \left\{ \begin{array}{l} \partial_x \phi_1 + i\sqrt{|\kappa|}\,\psi^* \phi_2 + i\zeta \phi_1 = 0 \\ \partial_x \phi_2 + i\sqrt{|\kappa|}\,\psi \phi_1 - i\zeta \phi_2 = 0 \end{array} \right\}$$

complex conj $\left\{ \begin{array}{l} \partial_x \phi_1^* - i\sqrt{|\kappa|}\,\psi \phi_2^* - i\zeta^* \phi_1^* = 0 \\ \partial_x \phi_2^* - i\sqrt{|\kappa|}\,\psi^* \phi_1^* + i\zeta^* \phi_2^* = 0 \end{array} \right\}$

So if $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ is a solution of ① with $\zeta \in \mathbb{C}$

then $\tilde{\phi} = \begin{pmatrix} \phi_2^* \\ -\phi_1^* \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} = i\sigma_2 \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix}$ is a sol. with $\zeta^* \in \mathbb{C}$

and $\tilde{\phi}$ satisfies $\partial_t \tilde{\phi} = (c^* + 2i\zeta^{*2}\sigma_3)\,\tilde{\phi}$

* **Note** that $\tilde{\tilde{\phi}} = i\sigma_2 \tilde{\phi}^* = i\sigma_2\, i\sigma_2 \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix}^* = -\phi$ (which is obviously again a solution)

- $L\phi = 0$ is a 2×2 linear system of 1st. order eqs.

$\Rightarrow$ we need to find 2 indep solutions

(just as for the original 2nd order Schrödinger eq. $\exists$ 2 indep sol.)

Consider 2 solutions $L\phi = 0 = L\omega$, first at different $\zeta_{1,2}$:

i) $$\left(\partial_x - i\sqrt{|\kappa|}(\psi^*\sigma_+ + \psi\sigma_-) + i\zeta_1\sigma_3\right)\phi = 0$$

ii) $$\left(\partial_x - i\sqrt{|\kappa|}(\psi^*\sigma_+ + \psi\sigma_-) + i\zeta_2\sigma_3\right)\omega = 0$$

recall $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\Rightarrow$ $\sigma_\pm^T = \sigma_\mp$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\Rightarrow \boxed{\partial_x(\omega^T i\sigma_2\phi)} = (\partial_x\omega^T) i\sigma_2\phi + \omega^T i\sigma_2\partial_x\phi$$

$$= i\sqrt{|\kappa|}(\psi^*\omega^T\sigma_- + \psi\omega^T\sigma_+) i\sigma_2\phi - i\zeta_2\omega^T\sigma_3 i\sigma_2\phi$$

$$+ \omega^T i\sigma_2 \, i\sqrt{|\kappa|}(\psi^*\sigma_+ + \psi\sigma_-)\phi - i\zeta_1\omega^T i\sigma_2\sigma_3\phi$$

$$= \boxed{i(\zeta_1 - \zeta_2)\omega^T\sigma_1\phi}$$ using $\sigma_-\sigma_2 = \begin{pmatrix} 0 & 0 \\ 0 & -i \end{pmatrix} = -\sigma_2\sigma_+$, $\sigma_+\sigma_2 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} = -\sigma_2\sigma_-$

in components we get

$$0 = \partial_x\Big(\omega^T \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\phi}_{\begin{pmatrix} \phi_2 \\ -\phi_1 \end{pmatrix}}\Big) - i(\zeta_1 - \zeta_2)\,\omega^T \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\phi}_{\begin{matrix} \phi_2 \\ \phi_1 \end{matrix}}$$

$$0 = \partial_x(\omega_1\phi_2 - \omega_2\phi_1) - i(\zeta_1 - \zeta_2)(\omega_1\phi_2 + \omega_2\phi_1)$$

Defining the analogue of the Wronskian

$$W(\omega, \phi) \equiv \omega^T i\sigma_2\phi = \omega_1\phi_2 - \omega_2\phi_1 = -W(\phi, \omega)$$

we have for equal spectral parameter $\zeta_1 = \zeta_2$ $\boxed{\partial_x W(\omega, \phi) = 0}$

standard Wronskian $W(f_1, \dots, f_n) \equiv \begin{vmatrix} f_1 & \dots & f_n \\ f_1' & & \\ \vdots & & \\ f_1^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$

f_i solutions of lin. diff. eq

$W \neq 0 \Rightarrow$ indep

$W(f, g) = f g' - f' g$. note that to linearise Schrödi we had $\phi_2 = \partial_x \phi_1$

The construction to find solutions using the Wronskian is called Jost functions.

From now on we consider the spectral parameter to be real $\xi \in \mathbb{R}$

Suppose we have 2 solutions $\left\{ \begin{matrix} L f = 0 = L g \\ \partial_t f = B f, \ \partial_t g = B g \end{matrix} \right\}$ with the following (plane wave) asymptotic

- $f(x, \xi) \xrightarrow[x \to +\infty]{} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i \xi x} \Rightarrow \tilde{f}(x, \xi) = \begin{pmatrix} f_2^* \\ -f_1^* \end{pmatrix} \xrightarrow[x \to +\infty]{} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x}$ is a solution
- $g(x, \xi) \xrightarrow[x \to -\infty]{} \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{-i\xi x} \Rightarrow \tilde{g}(x, \xi) = \begin{pmatrix} g_2^* \\ -g_1^* \end{pmatrix} \xrightarrow[x \to -\infty]{} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i \xi x}$

We furthermore assume that f and g ($\Rightarrow$ also $\tilde{f}, \tilde{g}$ is a solution) maintain their asymptotic form for large $\pm x$ for all times:

$\Rightarrow$ using the asymptotic form $\partial_t f = (c + 2i \xi^2 \sigma_3) f$ for $|x| \to \infty$ from p. 74

we have for $x \to +\infty$

$$\partial_t \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta x} \right) = 0 = \left(\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} + 2i\zeta^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta x}$$

$$0 \overset{!}{=} \begin{pmatrix} c+2i\zeta & 0 \\ 0 & c-2i\zeta^2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta x}$$

$\Rightarrow \quad c = 2i\zeta^2$

$$\Rightarrow \quad \underline{\frac{\partial f}{\partial t} = 2i\zeta^2 (\mathbb{1} + \sigma_3) f} \qquad \text{for } |x| \to \infty$$

ditto we can see for $\tilde{g}$ at $x \to -\infty$ $\quad c = 2i\zeta^2$

$$\underline{\frac{\partial}{\partial t} \tilde{g} = 2i\zeta^2 (\mathbb{1} + \sigma_3) \tilde{g}} \qquad \text{for } |x| \to \infty$$

and for $\tilde{f}$ at $x \to +\infty$ and g at $x \to -\infty$ $\quad c = -2i\zeta^2$

$$\underline{\frac{\partial}{\partial t} \tilde{f} = -2i\zeta^2 (\mathbb{1} - \sigma_3) \tilde{f}}$$

$$\underline{\frac{\partial}{\partial t} g = -2i\zeta^2 (\mathbb{1} - \sigma_3) g}$$

- because both $f, \tilde{f}$ and $g, \tilde{g}$ form an indep set of solutions ($f \perp \tilde{f}$ asymptot.)

we can expand one in terms of the other

$$\boxed{\begin{aligned} f(x,\zeta) &= a(\zeta)\, \tilde{g}(x,\zeta) + b(\zeta)\, g(x,\zeta) \\ \tilde{f}(x,\zeta) &= -\tilde{a}(\zeta)\, g(x,\zeta) + \tilde{b}(\zeta)\, \tilde{g}(x,\zeta) \end{aligned}}$$

with the coeffs $a, b, \tilde{a}, \tilde{b}$ being x-indep.

or in components
$$f_1 = a\tilde{g}_1 + b g_1 = a g_2^* + b g_1 \quad | \cdot g_2$$
$$f_2 = a\tilde{g}_2 + b g_2 = -a g_1^* + b g_2 \quad | \cdot g_1$$

- Using the fact that $f, g, \tilde{f}, \tilde{g}$ are normalised,

  i.e. $f_1 f_1^* + f_2 f_2^* = 1 = g_1 g_1^* + g_2 g_2^*$

  we can solve for the coefficient functions $a, b, \tilde{a}, \tilde{b}$:

  $\Rightarrow$ from above taking the difference we have

$$f_1 g_2 - f_2 g_1 = a(g_2 g_2^* + g_1 g_1^*) + b(g_1 g_2 - g_2 g_1)$$

$$\Longleftrightarrow \quad \boxed{W(f, g) = f_1 g_2 - f_2 g_1 = a(\zeta)}$$

similarly we have (exercise)

$$\boxed{\begin{aligned} -W(f, \tilde{g}) &= f_2 \tilde{g}_1 - f_1 \tilde{g}_2 = b(\zeta) \\ W(\tilde{f}, \tilde{g}) &= \tilde{f}_1 \tilde{g}_2 - \tilde{f}_2 \tilde{g}_1 = \tilde{a}(\zeta) \\ W(\tilde{f}, g) &= \tilde{f}_1 g_2 - \tilde{f}_2 g_1 = \tilde{b} \end{aligned}}$$

- the definition $\tilde{f} = i\sigma_2 f^* = \begin{pmatrix} f_2^* \\ -f_1^* \end{pmatrix}, \quad \tilde{g} = \begin{pmatrix} g_2^* \\ -g_1^* \end{pmatrix}$

  then implies (for $\zeta \in \mathbb{R}$) $\tilde{a} = a^*, \quad \tilde{b} = b^*$

- because the expansion (∇) of right asymptotic states into left asymptotic can be interpreted as transmission and reflection in the 1D case,

$$(\tilde{f}\ f) = \begin{pmatrix} \tilde{f}_1 & f_1 \\ \tilde{f}_2 & f_2 \end{pmatrix} = \begin{pmatrix} -g_1 & \tilde{g}_1 \\ -g_2 & \tilde{g}_2 \end{pmatrix} \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} =: (g\ \tilde{g})\, T^{-1}$$

with $T = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$, $\det T = 1$ unitary the time dependence of the coefficients $a = a(\varsigma, t)$ etc

provides us with the scattering data.

In detail we can use the asymptot. eqs as a, b, a^*, b^* are x-indep

- $$\frac{\partial}{\partial t} a(\varsigma, t) = \lim_{x\to\infty} \partial_t (f_1 g_2 - f_2 g_1)$$

$$= \lim_{x\to\infty} \partial_t (0 \cdot g_2 - f_2 g_1) = (\partial_t f_2 \, g_1 - f_2 \partial_t g_1)_{x\to\infty}$$

p. 78, valid at $|x|\to\infty$

$$= -\underbrace{\left(\partial_t \left(1 \cdot e^{i\varsigma x}\right)\right)}_{0} g_1\Big|_{x\to\infty} - 1 \cdot e^{i\varsigma x} (-2i\varsigma^2) \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} g \Big|_{\substack{1\text{-comp} \\ x\to\infty}} = 0$$

$$\Rightarrow \boxed{a(\varsigma, t) = a(\varsigma, 0)}, \text{ ditto for } *$$

- $$\frac{\partial}{\partial t} b(\varsigma, t) = \lim_{x\to\infty} \partial_t (f_2 \tilde{g}_1 - \underbrace{f_1}_{\to 0} \tilde{g}_2) = \lim_{x\to\infty} ((\underbrace{\partial_t f_2}_{\to 0}) g_1 + f_2 \partial_t \tilde{g}_1)$$

$$= f_2 \, 4i\varsigma^2 \tilde{g}_1$$

p78.

$$\partial_t \tilde{g} = 2i\varsigma^2 \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \tilde{g}$$

$$\Rightarrow \boxed{b(\varsigma, t) = e^{4i\varsigma^2 t} \, b(\varsigma, 0)}$$

- the time evolution of the bound state coeff. can be computed via analytic continuation of the $b(\varsigma, t)$.
- together with the corresponding matrix Gelfand-Levitan eq. this will provide the time dependent solution of NLS