Lax-Pair and Zero Curvature Condition

- recall the Lax pair in the 1st order formalism was parametr. as

$$\mathcal{L} = \mathbb{1}\partial_x - q(x,t)\sigma_+ - r(x,t)\sigma_- + i\zeta\sigma_3$$
$$\equiv \mathbb{1}\partial_x - A(x,t;\zeta) \qquad A \text{ 2x2 matrix}$$

$$B(x,t;\zeta) = P(x,t)\sigma_+ + Q(x,t)\sigma_- + R(x,t)\sigma_3 \qquad \text{2x2 matrix}$$

- the Lax equation relating $\mathcal{L}$ with B:

$$\partial_t \mathcal{L} = [B, \mathcal{L}]$$
$$\Leftrightarrow \quad [\partial_t - B, \mathcal{L}] = 0$$
$$\Leftrightarrow \quad [\partial_t - B, \partial_x - A] = 0$$
$$\cdot(-1) \qquad \boxed{\partial_t A - \partial_x B - [B, A] = 0}$$

* this equals the vanishing of a non-Abelian field-strength tensor in 2 dimensions: "zero curvature condition"

$$\boxed{\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \equiv F_{\mu\nu} = 0} \quad \mu,\nu = 0,1$$

when identifying $\quad A_0 \equiv -B(x,t;\zeta)$ and $\partial_t \equiv \partial_0$

$\qquad\qquad\qquad\quad A_1 \equiv -A(x,t;\zeta) \qquad \partial_x \equiv \partial_1$

- the fact that $F_{\mu\nu} = 0$ has implications on its solution as well as on the transmission matrix (p 79) $\quad T(\zeta)\underbrace{(\tilde{f}\ f)}_{F} = \underbrace{(-g\ \tilde{g})}_{G}$

Excursion local vs global symmetries & gauge transformations

- <u>global symmetry</u>:

  example: - multiplication of the complex wave function $\Psi(x)$ in the Schrödinger eq. (linear or non-linear) by a constant phase $\Psi(x,t) \to e^{i\chi}\Psi(x,t)$

  $\rightsquigarrow$ physical observables don't depend on χ $\qquad \chi \in \mathbb{R}$ const

- <u>local symmetry</u>: "gauging of a symmetry", making it space(-time) dependent

  example: $\Psi(x,t) \to e^{i\chi(x)}\Psi(x,t)$ is no longer a sym. of the Schrödinger eq as $\partial_x \Psi \to \partial_x e^{i\chi(x)}\Psi(x,t) \neq e^{i\chi(x)}\partial_x \Psi(x,t)$

<u>Q</u>: Can we construct an eq. that is invariant under such a local rotation?

<u>A</u>: Yes, by introducing a local gauge field $A_\mu(x)$:

transforming inhomogeneously:

$$\boxed{\begin{aligned} &A_\mu(x) \to A_\mu(x) + \partial_\mu \chi(x) \\ \text{when}\quad &\Psi(x) \to e^{i\chi(x)} \end{aligned}} \; *$$

$x = x^\mu,\ \mu = 0,1,\dots,d$ in any dimension

<u>and</u> replacing derivatives by covariant derivatives

$$\partial_\mu \to \partial_\mu - iA_\mu \equiv \mathcal{D}_\mu$$

• this only works for relativistic equations,

e.g.
$$\underline{i\gamma^\mu\left(\partial_\mu - iA_\mu(x)\right)\Psi(x) - m\Psi(x) = 0} \quad \text{Dirac eq.}$$

is invariant under the local trafo $\triangle$

• local observables depending on $|\Psi|^2$ still don't depend on $\underline{\underline{\chi(x)}}$

another example:
$$D_\mu D^\mu \phi(x) + m\phi(x) = 0 \quad \text{Klein Gordon eq.}$$

for complex field $\phi(x) \in \mathbb{C}$ with $\mathcal{L} = D_\mu\phi D_\mu\phi^* - m\phi\phi^*$

(in the non relativistic limit this gives the coupling of the gauge potential $A_\mu = (A_0, \vec{A})$ in the Schrödinger eq.
$$H = \frac{1}{2m}\left(\vec{p} - c\vec{A}\right)^2 + eA_0 - \frac{e}{2m}\vec{\sigma}\cdot\vec{A}$$
)

• in order to have $A_\mu(x)$ as a dynamical field we need to add a kinetic term for A_μ to the Lagrangian:
$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}$$
with $F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ the (Abelian) field strength

here A_μ, A_ν are complex valued fields so $[A_\mu, A_\nu] = 0$

• the e.o.m. for A_μ from $\partial^\beta \frac{\partial\mathcal{L}}{\partial\partial_\beta A^\alpha} - \frac{\partial\mathcal{L}}{\partial A^\alpha} = 0$ are $\underbrace{\partial^\mu F_{\mu\nu} = 0}_{\text{Maxwell}}$

- $F_{\mu\nu}$ is invariant under gauge trafo $A_\mu \to A_\mu + \partial_\mu \chi(x)$ (ex check!)
- the condition of vanishing curvature

$$\underline{F_{\mu\nu} = 0}$$

has an explicit solution $\underline{A_\mu(x) = \partial_\mu \Lambda(x)}$,

this is also called <u>pure gauge</u>.

<u>Non-Abelian Gauge Theory</u> (Yang-Mills Theory)

- the ideas above can be generalised to vector valued fields, e.g. $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \phi$
- the corresponding field strength $F_{\mu\nu}$ and gauge field A_μ are now matrix valued,

e.g. for 2vector ϕ: $\underline{A_\mu = A_\mu^a(x)\,\sigma_a}$, $a = 1,2,3$ Pauli-matrices

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

or $F_{\mu\nu} = F_{\mu\nu}^a \sigma_a = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + 2i\varepsilon^{abc} [A_\mu^b, A_\nu^c]) \cdot \sigma_a$

- the gauge trafo is now def as

$$\phi \to U\phi$$

$$A_\mu \to U A_\mu U^{-1} - i(\partial_\mu U)U^{-1}$$

with $U = U(x) = e^{i\sigma_a \chi^a(x)} \in SU(2) \Rightarrow U^{-1} = U^\dagger$

- the same concept of a covariant derivative D_μ action on now vector valued Dirac fields ψ or Klein-Gordon fields ϕ applies

- $F_{\mu\nu} = 0$ implies that

$$A_\mu(x) = (\partial_\mu U) U^{-1} \quad \text{is of pure gauge}$$

($\rightsquigarrow$ solution for $A(x), B(x)$ in 2d?)

- we will exploit another consequence of $F_{\mu\nu}=0$ related to Stoke's integral formula

<u>What is special about 2 dimensions?</u>

- in the 1st order formalism we parametrised A and B using σ_+, σ_- and σ_3 as a basis.

$\rightarrow$ these form a subalgebra of the (∞-dim)

<u>classical Virasoro algebra</u> $\boxed{[L_n, L_m] = (n-m) L_{n+m} \quad n,m \in \mathbb{Z}}$

- these are the <u>generators</u> of symmetry trafos in conformally invariant theories in 2D:
  - Poincaré-trafo (Lorenz + translat.)
  - scaling trafo
  - special conf. trafo

as $[\sigma_3, \sigma_\pm] = \pm 2\sigma_\pm$, $[\sigma_+, \sigma_-] = \sigma_3$

defining $L_0 = \frac{1}{2}\sigma_3$, $L_{+1} = -\sigma_-$, $L_{-1} = +\sigma_+$ $\Rightarrow$ $\boxed{\begin{aligned} [L_0, L_1] &= -L_1 \\ [L_0, L_{-1}] &= +L_{-1} \\ [L_{+1}, L_{-1}] &= 2L_0 \end{aligned}}$

Inverse scattering method via transition matrix

- we would like to formally solve

$$L\phi = 0 \iff \boxed{\partial_x \phi = A(x,\zeta)\,\phi}$$

$A = -L_1$
$B = -L_0$

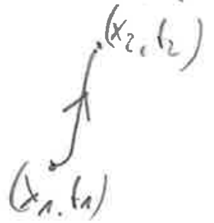
using a transition matrix $T(x,y;\zeta)$

s.th. $\boxed{\phi(x) = T(x,y;\zeta)\,\phi(y)}$

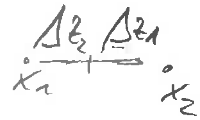
- after defining it properly and investigating its properties we will show that the scattering matrix $S(\zeta)$ p79 can be expressed through it ($\Rightarrow$ its t-dependence)

Consider $U_{\gamma_1}(x_2,t_2;x_1,t_1) = P_{\gamma_1}\exp\left[-\int_{(x_1,t_1)}^{(x_2,t_2)} dx^\mu A_\mu(x,t)\right]$

where curve γ_1 connects the points



and P_{γ_1} is the path ordered product.



e.g. $x_1 \approx x_2$: $\int_{x_1,\gamma}^{x_2} dz\,A(z) = \Delta z_1 A(z_1) + \Delta z_2 A(z_2)$

$\Rightarrow P_{\gamma_1}$ put later parts on the curve left

$$P_{\gamma_1}\exp[-\Delta z_1 A(z_1) - \Delta z_2 A(z_2)] = \mathbb{1} - (\Delta z_1 A(z_1) + \Delta z_2 A(z_2)) + \frac{1}{2}\left(\Delta z_1^2 A(z_1)^2 + 2\Delta z_1 \Delta z_2 A(z_2)A(z_1) + \Delta z_2^2 A(z_2)^2\right) + \dots$$

properties:

$\int_\gamma dx^\mu A_\mu$ is independent of the path, or in other words

$\oint_{\mathcal{C}\text{ closed}} dx^\mu A_\mu = 0$

Consider 2 curves with common end/start

$\gamma_1 \cup \gamma_2 = \mathcal{C}$

$$\Rightarrow \quad \int_{\gamma_1} dx^\mu A_\mu(x) + \int_{\gamma_2} dx^\mu A_\mu(x) = \int_{\mathcal{C}} dx^\mu A_\mu(x)$$

$$\overset{\text{Stokes}}{=} \underbrace{\int da\, e_\mu \epsilon^{\mu\nu\rho} \partial_\nu A_\rho(x)}_{\text{area enclosed by } \mathcal{C}} = \frac{1}{2} \underbrace{\int da\, e_\mu \epsilon^{\mu\nu\rho} (\partial_\nu A_\rho - \partial_\rho A_\nu)}_{\text{area of } \mathcal{C}}$$

for an Abelian field strength with zero curvature $F_{\nu\rho} = 0$ this vanishes and is thus path indep.!

- In particular $\int_{\mathcal{C}} dx^\mu A_\mu$ is gauge invariant (here $F_{\nu\rho} = 0$ not needed)

the object $U_{\mathcal{C}\,(\gamma)}$ is called <u>Wilson loop</u> (line).

$\Rightarrow$ properties of U: for the above γ_1, γ_2 we have

- $U_{\gamma_1}(x_2, t_2; x_1 t_1)\, U_{\gamma_2}(x_1 t_1; x_2 t_2) = e^{\int_{\gamma_1} dx^\mu A_\mu}\, e^{\int_{\gamma_2} dx^\mu A_\mu} = e^0 = 1$

$\uparrow$ Abelian: $e^A e^B = e^{A+B} \underset{\uparrow}{+} \frac{1}{2}[A, B]$

else Baker-Campbell-Hausdorff

$\Rightarrow$

- $U_{\gamma_1}(x_2 t_1; x_1 t_1)^{-1} = U_{-\gamma_1}(x_1 t_1; x_2 t_2)$

$U_\gamma(x\, t; x, t) = 1$

Because of the path independence we can choose the path along 1-direction $T(x, y, \xi) \equiv U(x, t; y, t) = P e^{-\int_y^x dz A_1(z)} = P e^{+\int_x^y dz A(z, \xi)}$

The transmission matrix T inherits the properties of U:

$$T^{-1}(x,y;\xi) = T(y,x;\xi) \qquad T(x,x;\xi) = \mathbb{1}$$

$$T(x,y;\xi)\,T(y,z;\xi) = T(x,z;\xi)$$

From the path ordering we have $\qquad \int_y^x$

$$\partial_x T(x,y;\xi) = A(x,\xi)\,T(x,y;\xi)$$

$$\partial_y T(x,y;\xi) = -T(x,y;\xi)\,A(y,\xi)$$

and it solves $\partial_x \phi(x) = A(x,\xi)\,\phi(x)$ by $\phi(x) = T(x,y;\xi)\,\phi(y)$

as $\partial_x \underbrace{T(x,y;\xi)\phi(y)}_{\phi(x)} = A(x,\xi)\,T(x,y;\xi)\,\phi(y) = A(x,\xi)\,\phi(x)$

asymptotic analysis:

from $A(x,\xi) = i\sqrt{|\kappa|}\,(\psi^* \sigma_+ + \psi \sigma_-) - i\xi\sigma_3$

we have $\quad A(x,\xi) = -i\xi\sigma_3 \quad$ for $\psi(x), \psi^*(x) = 0$ (which is true at $|x| \to \infty$)

$$\Rightarrow \quad T_0(x,y;\xi) = e^{-i\xi\sigma_3(x-y)} = T_0(x-y;\xi)$$

We also have for $x,y \to +\infty \; (-\infty)$ $\quad \lim_{x,y\to +\infty\,(-\infty)} \int_y^x dz\, A(z,\xi) = -i\xi\sigma_3(x-y)$

$$\Rightarrow \quad \lim_{x,y \to \pm\infty} T(x,y;\xi) = e^{-i\xi\sigma_3(x-y)} = T_0(x-y;\xi)$$

defining
$$\boxed{\begin{aligned} F(x,\xi) &\equiv \lim_{y\to\infty} T(x,y;\xi)\,T_0(y;\xi) \\ G(x,\xi) &\equiv \lim_{y\to-\infty} T(x,y;\xi)\,T_0(y;\xi) \end{aligned}}$$

we have
$$\lim_{x\to\infty} F(x,\xi) = \lim_{x,y\to\infty} T(x,y;\xi)T_0(y;\xi) = e^{-i\xi\sigma_3(x-y)}\,e^{-i\xi\sigma_3 y}$$

likewise
$$= \begin{pmatrix} e^{-i\xi x} & 0 \\ 0 & e^{i\xi x}\end{pmatrix} = \lim_{x\to+\infty}\begin{pmatrix} | & | \\ \tilde f & f \\ | & | \end{pmatrix}$$

and ditto
$$\lim_{x\to-\infty} G(x,\xi) = \begin{pmatrix} e^{-i\xi} & 0 \\ 0 & e^{i\xi}\end{pmatrix} = \lim_{x\to-\infty}\begin{pmatrix} | & | \\ -g & \tilde g \\ | & | \end{pmatrix}$$

Because we have
$$(\tilde f\, f)\,T(\xi) = (-g\,\tilde g)\,, \quad T(\xi) = \begin{pmatrix} a & b \\ -b^* & a^*\end{pmatrix}$$
$$F\,T(\xi) = G \iff \underline{T(\xi) = F^{-1}G}$$

we can represent the scattering matrix by Wilson lines

$$T(\xi) = \lim_{\substack{y\to\infty \\ z\to-\infty}} T_0(y,\xi)^{-1}\,\underbrace{\underbrace{T(x,y;\xi)^{-1}}_{T(y,x;\xi)}\,T(x,z;\xi)}_{T(y;z,\xi)}\,T_0(z,\xi)$$

$$= \lim_{\substack{y\to+\infty \\ z\to-\infty}} e^{i\xi\sigma_3 y}\,T(y,z;\xi)\,e^{-i\xi\sigma_3 z}$$

- from this the same time evolution for $a(\xi,t)$, $b(\xi,t)$ can be deduced as on p. 80, using $\partial_t T(\xi) = 2i\xi^2[\sigma_3, T(\xi)]$ (exercise)