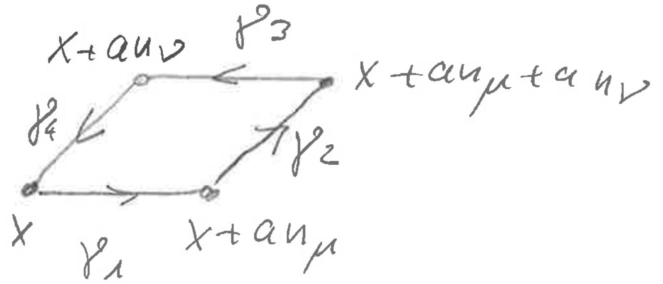
* Non-Abelian Wilson lines

Here we will show that infinitesimally the product of Wilson lines forming a closed loop equals the infinitesimal loop-surface integral of the non-Abelian field strength

- for the path we choose



- where n_ν is the unit vector in ν-direction
- a is the length of each path

- for an integral over an infinitesimal segment we have

$$\int_x^{x+\Delta x} ds\, f(s) \approx \Delta x\, f(x) \approx \Delta x\, f(x+\Delta x) \approx \Delta x\, f(x+\tfrac{1}{2}\Delta x)$$

end points   mid point

$\Rightarrow$ we get for the 4 directed integrals with $a \ll 1$

$$I_1 = \int_{\gamma_1} ds^S A_S = \int_x^{x+an_\mu} ds^S A_S \approx a A_\mu(x+\tfrac{a}{2}n_\mu)$$

$$I_2 = \int_{\gamma_2} ds^S A_S = \int_{x+an_\mu}^{x+an_\mu+an_\nu} ds^S A_S \approx a A_\nu(x+an_\mu+\tfrac{a}{2}n_\nu)$$

$$I_3 = \int_{\gamma_3} ds^S A_S = \int_{x+an_\mu+an_\nu}^{x+an_\nu} ds^S A_S \approx -a A_\mu(x+an_\nu+\tfrac{a}{2}n_\mu)$$

$$I_4 = \int_{\gamma_4} ds^S A_S = \int_{x+an_\nu}^{x} ds^S A_S \approx -a A_\nu(x+\tfrac{a}{2}n_\nu)$$

inserting these in $U_{\gamma_1}(x+a u_\mu, x) = P_{\gamma_1} e^{[-\int_{\gamma_1} ds^g A_g]}$

- we can drop the path order
- the exponent $a A_\mu(x+\frac{a}{2}u_\mu) = a A^b_\mu(x+\frac{a}{2}u_\mu)\sigma_b$ remains a non-commuting 2x2 matrix

$\Rightarrow$ consider $\boxed{e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots}}$ ← breaks off off $[A,B] \sim 1$

Baker-Campbell-Hausdorf

- here $A, B = -I_j = \mathcal{O}(a)$, so

$$e^{-I_2} e^{-I_1} = e^{-I_2 - I_1 - [I_2, I_1] + \mathcal{O}(a^3)}$$

$$\Rightarrow e^{-I_4} e^{-I_3} e^{-I_2} e^{-I_1} = e^{-\sum_{j=1}^{4} I_j + \frac{1}{2}\sum_{j>i}[I_j, I_i] + \mathcal{O}(a^3)}$$

↑ exercise ✓

Claim: $\boxed{U_{\gamma_4} U_{\gamma_3} U_{\gamma_2} U_{\gamma_1} \stackrel{!}{=} e^{-a^2 F_{\mu\nu}}}$ to leading order

taking log $\Rightarrow \underbrace{-\sum_{j=1}^{4} I_j} + \frac{1}{2}\sum_{j>i}[I_j, I_i]$

$$= -a\left(A_\mu(x+\tfrac{a}{2}u_\mu) + A_\nu(x+a u_\mu + \tfrac{a}{2}u_\nu) - A_\mu(x+a u_\nu + \tfrac{a}{2}u_\mu) - A_\nu(x+\tfrac{a}{2}u_\nu)\right)$$

$$= -a\left(-a\partial_\nu A_\mu(x) + a\partial_\mu A_\nu(x)\right) + \mathcal{O}(a^3)$$

$$+\frac{1}{2}\sum_{j>i}[I_j, I_i] = +\frac{a^2}{2}\left([-A_\nu(x), -A_\mu(x)] + [-A_\nu, A_\mu] + [-A_\mu, A_\nu] + [A_\nu, A_\mu]\right) + \mathcal{O}(a^3)$$

$$= +a^2[A_\nu, A_\mu]$$

Conformal Symmetry

- the conformal group of symmetry trafo generalises the Poincaré trafos (translations, rotations, Lorenz-trafos)
- although conformal symmetry of a theory has important bearings in any dimension d (3&4-point fnct. β-function) we will see why $d=2$ is special

Consider a general infinitesimal coordinate trafo

$$x^\mu \to \boxed{x'^\mu = x^\mu + \underbrace{\epsilon^\mu(x)}_{\text{space-time dependent}} + \mathcal{O}(\epsilon^2)}$$

with $\mu = 0, 1, \dots, d-1$
$x^0 =$ time t
d dim. of space-time

example: translations

$$\underline{x'^\mu = x^\mu + \epsilon^\mu} \quad , \epsilon^\mu = \text{const}$$

tensor transformation law under general coord. trafo

$$\phi_{\mu_1 \dots \mu_n}(x) \to \phi'_{\mu_1 \dots \mu_n}(x') = \frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial x^{\nu_n}}{\partial x'^{\mu_n}} \phi_{\nu_1 \dots \nu_n}(x)$$

tensor of n-th level

examples:
- scalar $\phi(x) \to \phi'(x') = \phi(x)$ doesn't transform (0-th level)
- tensor 2-nd level e.g. metric

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$$

$$\equiv g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$$

general transformation:

$$\frac{\partial x^\rho}{\partial x'^\mu} = \frac{\partial}{\partial x'^\mu}\left(x'^\rho - \epsilon^\rho(x)\right) = \delta^\rho_\mu - \partial'_\mu \epsilon^\rho(x) \left(+ \mathcal{O}(\varepsilon^2)\right)$$

$$\Rightarrow g'_{\mu\nu}(x') = \left(\delta^\rho_\mu - \partial'_\mu \epsilon^\rho\right)\left(\delta^\sigma_\nu - \partial'_\nu \epsilon^\sigma\right) g_{\rho\sigma}(x)$$

drop

$$= g_{\mu\nu}(x) - \left((\partial'_\mu \epsilon^\rho)\delta^\sigma_\nu + \delta^\rho_\mu (\partial'_\nu \epsilon^\sigma)\right) g_{\rho\sigma} + \mathcal{O}(\varepsilon^2)$$

$$\Rightarrow \delta g_{\mu\nu}(x) = -\partial_\mu \epsilon^\rho(x)\, g_{\rho\nu} - \partial_\nu \epsilon^\sigma g_{\mu\sigma} + \mathcal{O}(\varepsilon^2)$$

$$\underline{\delta g_{\mu\nu} = -\left(g_{\rho\nu}\partial_\mu + g_{\rho\mu}\partial_\nu\right)\epsilon^\rho(x)}$$

using $g_{\mu\nu} = g_{\nu\mu}$ (for flat Minkowski space we have $g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1\!\!1_{d-1}\end{pmatrix}$)

<u>example:</u> translations $\varepsilon^\mu = \text{const}$ $= \frac{\partial x^\rho}{\partial x'^\nu} = \delta^\rho_\nu$

$\Rightarrow \delta g_{\mu\nu} = 0$ i.e. $g'_{\mu\nu}(x') = g_{\mu\nu}(x)$

ditto $v_\mu(x) \to v'_\mu(x') = \delta^\rho_\mu v_\rho(x) = v_\mu(x)$

for vectors

* translations don't change the scalar product

$v \cdot w = v^\mu w^\nu g_{\mu\nu} = v^\mu w_\mu$ is invariant

* also rotations and Lorentz trafo's don't change $v \cdot w$

a particular class of infinitesimal trafos is given by

- <u>Weyl-transformations</u>: ("rescaling")

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x)\, g_{\mu\nu}(x) \quad \text{on the metric}$$

or infinitesimally: $\Omega = 1 + \omega(x)$: $\quad g'_{\mu\nu}(x') = g_{\mu\nu}(x) + \underbrace{\omega(x)\, g_{\mu\nu}(x)}_{\delta g_{\mu\nu}(x)}$

$|\omega(x)| \ll 1$

$\omega > 0$ inflating, $\omega < 0$ shrinking

* <u>Weyl trafos preserves angles</u>:

$$v \cdot w \to \Omega(x)\, v \cdot w$$

and $v \cdot w = \|v\|\,\|w\| \cos\theta$

where $\|v\|^2 = v \cdot v$

$$\Rightarrow \quad \frac{v \cdot w}{\|v\|\,\|w\|} = \frac{v \cdot w}{\sqrt{v^2\, w^2}} = \cos\theta \quad \text{is invariant}$$

<u>Definition</u>: <u>conformal transformation</u> in d dimensions

= all coordinate trafo's acting on the metric like a Weyl trafo (e.g. translations: $\Omega = 1$ or $\omega(x) = 0$) :

$$\delta g_{\mu\nu} = -\left(g_{\rho\nu}\partial_\mu - g_{\rho\mu}\partial_\nu\right)\epsilon^\rho(x) \overset{!}{=} \omega(x)\, g_{\mu\nu}$$

$g_{\mu\nu}$ metric: $\quad g^{\sigma\mu} g_{\mu\nu} = \delta^\sigma_\nu \quad , \quad a^\rho = g^{\rho\mu} a_\mu$

$$\cdot g^{\mu\alpha} g^{\nu\beta} \Big| \quad -\left(\delta^\beta_\rho \partial^\alpha + \delta^\alpha_\rho \partial^\beta\right)\epsilon^\rho(x) = g^{\alpha\beta}\omega(x)$$

$$\Leftrightarrow \quad \underline{-\partial^\alpha \epsilon^\beta(x) - \partial^\beta \epsilon^\alpha(x) = \omega(x)\, g^{\alpha\beta}}$$

using $\mathrm{Tr}\, g^2 = g_{\alpha\beta} g^{\beta\alpha} = \delta_\alpha^\alpha = d$

and contracting with $g_{\alpha\beta}$ we obtain

$$-\partial_\beta \epsilon^\beta - \partial_\alpha \epsilon^\alpha = \omega(x) \cdot d \iff \boxed{\omega(x) = \frac{2}{d}(-)\partial_\beta \epsilon^\beta(x)}$$

$$\Rightarrow \boxed{\partial^\alpha \epsilon^\beta(x) + \partial^\beta \epsilon^\alpha(x) = \frac{2}{d}\left(\partial_\varrho \epsilon^\varrho(x)\right) g^{\alpha\beta}} \quad (*) \qquad \frac{2}{d}\, \overset{4}{\partial} \cdot \epsilon$$

all transformations $\epsilon^\mu(x)$ that satisfy this eq. define <u>conformal trafos!</u>

- contracting with $\partial_\sigma \partial_\alpha$ and assuming all derivatives commute we have

$$\partial_\sigma \partial_\alpha \partial^\alpha \epsilon^\beta + \partial_\sigma \partial^\beta \partial_\alpha \epsilon^\alpha = \frac{2}{d} \partial_\sigma \partial^\beta \partial_\varrho \epsilon^\varrho$$

$$\iff \boxed{\partial_\sigma \partial_\alpha \partial^\alpha \epsilon^\beta + \left(1 - \frac{2}{d}\right) \partial_\sigma \partial^\beta \partial_\varrho \epsilon^\varrho = 0}$$

$\Rightarrow$ we have to distinguish $d=2$ (see below) and $d>2$

$\left(d=1: \; g^{\alpha\beta}(x) = g^{00}(x) \,, \quad g^{00}\partial_0 = \partial^0 \,, \quad (*) \text{ trivially satisfied } \forall\, \epsilon^0(x) \right)$

<u>Conformal transformations in $d>2$ dimensions</u> :

- one can show for $d>2$ that it holds

$$\partial^\mu \partial^\nu \partial^\varrho \epsilon^\sigma(x) = 0 \qquad \text{(exercise)}$$

$\Rightarrow$ $\epsilon^\mu(x)$ is at most quadratic in x^α :

<u>Ansatz</u> $\quad \underline{\epsilon^\mu(x) = \alpha^\mu + \beta^\mu{}_\nu x^\nu + \gamma^\mu{}_{\nu\varrho} x^\nu x^\varrho}$

$\Rightarrow$ determine α, β, γ's

The conformal group in $d>2$ dimensions

1) translations $\qquad x^\mu \to x^\mu + \alpha^\mu$

2) rotations and Lorenz trafos $\qquad x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu$

with $\omega_{\mu\nu} = -\omega_{\nu\mu}$ anti-sym.

} Poincaré trafos

3) resaling $\qquad x^\mu \to x^\mu + \sigma x^\mu$

4) special conformal trafo

$$x^\mu \to x^\mu + b^\mu x\cdot x - 2x^\mu b\cdot x$$
$$= x^\mu + b^\mu x_\alpha x^\alpha - 2b^\alpha x_\alpha x^\mu$$

these are all infinitesimal conf. trafos that are quadratic in x^μ

- the <u>generators</u> of these trafos are

1) $P_\mu = \partial_\mu$ $\qquad$ momentum

2) $M_{\mu\nu} = \frac{1}{2}(x_\mu\partial_\nu - x_\nu\partial_\mu)$ $\qquad$ "angular momentum"

3) $D = x^\mu \partial_\mu$

4) $K_\mu = x\cdot x\,\partial_\mu - 2x_\mu x^\nu \partial_\nu$

to be contracted with $\alpha^\mu, \omega^\mu{}_\nu, \sigma, b^\mu$

Why? <u>example</u> translation $\quad f(x) \to f(x+\alpha) = f(x) + \alpha^\nu \underbrace{\partial_\nu}_{P_\nu \text{ generator}} f(x) + \mathcal{O}(\alpha^2)$

or rescaling $\quad f(x) \to f(x+\sigma x) = f(x) + \sigma \underbrace{x^\mu \partial_\mu}_{D \text{ generator}} f(x) + \mathcal{O}(\sigma^2)$