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Symmetries in Physics – Exercise Sheet 1

Exercise 1.1: Consider the action

$$S(t_0, t_1) = \int_{t_0}^{t_1} dt \left(\frac{1}{2} m \dot{\mathbf{q}}^T \dot{\mathbf{q}} + V(\mathbf{q}^T \mathbf{q}) \right), \tag{1}$$

where m is a constant, $\mathbf{q}$ a D-dimensional real vector with components $\mathbf{q}^T = (q_1, \ldots, q_D)$, $(.)^T$ is the transposition and a dot denotes the time derivative.

(i) Find the equations of motion and explain what system this action describes.

(ii) Find the conserved current associated with the transformation $t \to t' = t + a$. Explain its physical significance.

(iii) For which potential V is the associated current to the transformation $\mathbf{q}(t) \to \mathbf{q}'(t) = \mathbf{q}(t) + \mathbf{b}$ conserved? Explain its physical significance.

(iv) Show that Eq. (1) is invariant under the transformation $\mathbf{q}(t) = R\mathbf{q}(t)$ where R is an orthogonal matrix. Explain its physical significance. If we write an infinitesimal version of this transformation as $R = \mathbb{1}_D + \epsilon Q + O(\epsilon^2)$, what are the properties of the matrix Q?

(v) Verify the following commutation relations for the generators of translations ∂_j and of rotations $M_{ij} = q_i \partial_j - q_j \partial_i$:

$$[\partial_k, M_{ij}] = \delta_{ki} \partial_j - \delta_{kj} \partial_i$$

$$[M_{ij}, M_{kl}] = \delta_{jk} M_{il} - \delta_{jl} M_{ik} - \delta_{ik} M_{jl} + \delta_{il} M_{jk}$$

(vi) From now on the dimension should be $D = 3$. Show that under the transformation of part (iv) the conserved current is

$$\mathbf{j} = m \mathbf{q} \times \dot{\mathbf{q}}. \tag{2}$$

where $\times$ is the exterior (cross) product in three dimensions.

Exercise 1.2: Consider the action

$$S = \int_{\mathcal{V}} d^4x \left(\nabla^T \phi(\mathbf{x}) \eta \nabla \phi^*(\mathbf{x}) + V(\phi(\mathbf{x}) \phi^*(\mathbf{x})) \right), \tag{3}$$

where $\phi(x)$ is a scalar, complex field on a four-dimensional space-time volume $\mathcal{V}$, $\eta = \mathrm{diag}(1, 1, 1 - 1)$ is the flat Minkowski metric, $\nabla^T = (\partial_1, \partial_2, \partial_3, \partial_4)$ is the gradient and V is a finite polynomial.

(i) Show that the current corresponding to the translation $\mathbf{x} \to \mathbf{x}' = \mathbf{x} + \mathbf{b}$ (also known as the canonical energy-momentum tensor T) is conserved and is a real symmetric 4×4 matrix.

(ii) Show that the transformation $\phi(x) \to e^{\imath\Lambda} \phi(x)$, where $\Lambda \in \mathbb{R}$ is a constant, is a global symmetry and find the associated conserved current. Is the local transformation

$$\phi(x) \to e^{\imath\Lambda(x)} \phi(x) \tag{4}$$

a symmetry of the action (5)?

(iii) Consider a new Lagrangian

$$S = \int_{\mathcal{V}} d^4x \left(\mathcal{D}^T \phi(x) \eta \left(\mathcal{D}\phi(x)\right)^* + V(\phi(x)\phi^*(x))\right) \tag{5}$$

with the covariant derivative $\mathcal{D} = \nabla + \imath \mathbf{A}(x)$. How has the vector field $\mathbf{A} = \{A_\mu\}$ to transform such that the transformation (4) is a symmetry of this new action?

(iv) Show that the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is invariant under the transformation of $\mathbf{A}$ derived in (iii) and the action

$$S_{\mathrm{EM}} = \int d^4x \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{6}$$

is gauge invariant.

(v) Show that the canonical energy-momentum tensor of the action (6) takes the form

$$T^{\mu\nu}_{\mathrm{EM}} = F^{\mu\lambda} \partial^\nu A_\lambda - \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}. \tag{7}$$

Advice: Show first that $\partial \mathcal{L} / \partial(\partial_\sigma A_\lambda) = F^{\sigma\lambda}$.

(vi) Prove that when adding the term $F^{\lambda\mu}\partial_\lambda A^\nu$ to $T^{\mu\nu}_{\mathrm{EM}}$ the improved energy-momentum tensor is symmetric. Discuss in what situations it is permissible to add this term.