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Symmetries in Physics – Exercise Sheet 2

Exercise 2.1 Let the metric be the Minkowski metric, i.e. $g_{\mu\nu} = \eta_{\mu\nu}$. Consider the infinitesimal transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$ where $\epsilon^\mu(x)$ is a smooth vector-valued function with an infinitesimal amplitude. The definition of a conformal transformation is equivalent to

$$\partial_\nu \epsilon_\mu(x) + \partial_\mu \epsilon_\nu(x) = -\omega(x)\eta_{\mu\nu}. \quad (1)$$

Recall that from this equation another relation follows

$$2\partial_\mu \partial_\nu \epsilon_\rho(x) = \eta_{\mu\nu}\partial_\rho \omega(x) - \eta_{\mu\rho}\partial_\nu \omega(x) - \eta_{\nu\rho}\partial_\mu \omega(x) \quad (2)$$

leading to the constraint

$$(d-2)\partial_\mu \partial_\nu \omega(x) = -\partial^\rho \partial_\rho \omega(x)\eta_{\mu\nu}. \quad (3)$$

In more than two dimensions ($d > 2$) this equation can be readily integrated to the general solution $\omega(x) = A + B_\mu x^\mu$, where A and B^μ are constants.

(i) Show the general solution of the infinitesimal deformation solves the equation

$$\partial_\nu \partial_\lambda \partial_\kappa \epsilon_\mu(x) = 0 \quad (4)$$

and thus is

$$\epsilon_\mu(x) = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho. \quad (5)$$

and give a relation with the constants A and B_μ. Explain why we can choose $c_{\mu\nu\rho} = c_{\mu\rho\nu}$.

(ii) Show that the tensor $c_{\mu\nu\rho}$ can be simplified to

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu. \quad (6)$$

What is the relation between b_ν and B_ν?

(iii) Let $a_\mu = 0$ and $c_{\mu\nu\rho} = 0$ in Eq. (5). Show that

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d}b^\lambda_\lambda \eta_{\mu\nu}, \quad \text{and} \quad b_{\mu\nu} = m_{\mu\nu} + \alpha\eta_{\mu\nu}, \quad (7)$$

where m is an anti-symmetric tensor. What is the physical interpretation of α and $m_{\mu\nu}$?

(iv) Let us assume $a_\mu = 0$ and $b_{\mu\nu} = 0$ in Eq. (5), i.e. $\epsilon^\mu(x)$ is purely a quadratic function. Show that the infinitesimal transformation is

$$x^\mu \to x'^\mu = x^\mu + 2(x^\alpha b_\alpha)x^\mu - b^\mu x^\alpha x_\alpha \quad (8)$$

and its finite version is

$$x'^\mu = \frac{x^\mu - b^\mu x^\nu x_\nu}{1 - 2b^\nu x_\nu + b^\lambda b_\lambda x^\nu x_\nu}. \quad (9)$$

(v) Proof the equivalence of Eq. (9) with

$$\frac{x'^{\mu}}{x'^{\nu}x'_{\nu}} = \frac{x^{\mu}}{x^{\nu}x_{\nu}} - b^{\mu} \tag{10}$$

and geometrically interpret this transformation.

Exercise 2.2 Consider a two dimensional Euclidean space, i.e it has the metric $g_{\mu\nu} = \delta_{\mu\nu}$, with coordinates labelled (z_0, z_1). Moreover, consider a conformal transformation $z^{\mu} \to w^{\mu}(z)$ such that the metric transforms as,

$$\delta^{\mu\nu} \to (\partial_{\alpha} w^{\mu}(z))(\partial_{\beta} w^{\nu}(z))\delta^{\alpha\beta} = \Omega(z)\delta^{\mu\nu}. \tag{11}$$

(i) Show that Eq. (11) implies

$$\left(\frac{\partial w^0}{\partial z^0}(z)\right)^2 + \left(\frac{\partial w^0}{\partial z^1}(z)\right)^2 = \left(\frac{\partial w^1}{\partial z^0}(z)\right)^2 + \left(\frac{\partial w^1}{\partial z^1}(z)\right)^2, \tag{12}$$

$$\frac{\partial w^0}{\partial z^0}(z)\frac{\partial w^1}{\partial z^0}(z) = -\frac{\partial w^0}{\partial z^1}(z)\frac{\partial w^1}{\partial z^1}(z). \tag{13}$$

(ii) Show that Eq. (12) is equivalent to

$$\frac{\partial w^0}{\partial z^0}(z) = -\frac{\partial w^1}{\partial z^1}(z),\ \frac{\partial w^1}{\partial z^0}(z) = \frac{\partial w^0}{\partial z^1}(z), \quad \text{or} \quad \frac{\partial w^0}{\partial z^0}(z) = \frac{\partial w^1}{\partial z^1}(z),\ \frac{\partial w^1}{\partial z^0}(z) = -\frac{\partial w^0}{\partial z^1}(z). \tag{14}$$

When combining both coordinates to a complex variable $z = z_0 + iz_1$, show that any map $z \to f(z)$ is conformal if and only if f is meromorphic.

(iii) Infinitesimally such transforms have always the form $z \to z' = z + \sum_{n=-\infty}^{\infty} c_n z^{n+1}$ where the fields are transformed like $\phi(z) \to \phi'(z') = \phi(z(z'))$. Show that generators of theses transformations take the form, $L_n = -z^{n+1}\partial_z$. Moreover, show that they fulfill the classical Virasoro algebra,

$$[L_n, L_m] = (n-m)L_{n+m}. \tag{15}$$

Exercise 2.3 Here we want to recall some group properties. With help the group axioms, prove for a group G:

(i) The identity element is unique.

(ii) For each $a \in G$ there exists a unique left and right inverse which is the same, i.e. $a^{-1}a = aa^{-1} = e$.

Let the group G be a finite group. Then show additionally:

(iii) Each row and column of the multiplication table of G contains every element once.

(iv) Find all groups of order three.