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Symmetries in Physics – Exercise Sheet 3

Exercise 3.1

An application in classical Condensed Matter Theory: Let us consider a two dimensional lattice and each elementary cell is an equal-sided polygon with n vertices and n edges.

(i) Show that the only polygons of which one can construct a two-dimensional lattice are either equal-sided triangles ($n = 3$), squares ($n = 4$) or equal sided hexagons ($n = 6$). To prove this statement show first that the inner angle, φ, of an equal-sided polygon with n vertices is $\varphi = \pi(n-2)/n$.

(ii) Explain via the ratio of the volumes, V, of the polygons (triangle, square and hexagon) to their circumferences, c, that the hexagon has the optimal ratio, meaning V/c^2 is maximal. For this purpose calculate this ratio for an arbitrary equal-sided polygon. Explain what the physical impact is (consider some examples in nature).

(iii) The three types of polygons are symmetric under the dihedral group D_3 (triangle), D_4 (square), and D_6 (hexagon). Explain what the action of these groups on the polygon and thus on the lattice are and prove that the corresponding lattices are indeed invariant under these group actions.

(iv) How many elements do the groups D_3, D_4, and D_6 contain? Write their multiplication tables and interpret them physically.

(v) Show that D_3 is a subgroup of D_6 and interpret this as a group action on the two-dimensional lattice.

Exercise 3.2

An application in Quantum Mechanics: Let us consider a one-particle Hamilton operator,

$$H = -\Delta + V(x), \tag{1}$$

in two dimension where the potential V is invariant under the dihedral group D_n. This Hamiltonian models a one-particle system in an external potential. Let the center of the associated polygon and thus of the corresponding potential V be at the origin. One of its vertices should be located on the positive side of the x-axis.

(i) Show that each element of the dihedral group D_n can be generated by the rotation operator $\widehat{O}_n$ (rotating counter-clockwise with an angle $2\pi/n$ around the origin) and the reflection, $\widehat{R}$, at the x-axis, i.e.

$$g \in D_n \Leftrightarrow \left(\exists! m \in \{1,\ldots,n\} \text{ and } l \in \{0,1\} \text{ such that } g = \widehat{O}_n^m \widehat{R}^l\right). \tag{2}$$

(ii) What is the "commutator" of $\widehat{O}_n$ and $\widehat{R}$? In particular, show that

$$\widehat{R}\widehat{O}_n\widehat{R}^{-1} = \widehat{O}_n^{-1}. \tag{3}$$

(iii) Let $|\psi\rangle$ be an eigenfunction of H, i.e.

$$H|\psi\rangle = E|\psi\rangle. \tag{4}$$

Show that

$$|\psi_k\rangle = \sum_{l=1}^{n} e^{2\pi\imath lk/n} \widehat{O}_n^l |\psi\rangle, \quad k \in \{1, \ldots, n\} \tag{5}$$

and

$$|\phi_\pm\rangle = (1 \pm \widehat{R})|\psi\rangle \tag{6}$$

are also eigenfunctions of H. Moreover show the orthogonality relations:

$$\langle \psi_k | \psi_{k'} \rangle = 0, \quad k \neq k' \in \{1, \ldots, n\} \tag{7}$$

and

$$\langle \psi_+ | \psi_- \rangle = 0. \tag{8}$$

What are the eigenvalues of $\widehat{O}_n$ for $|\psi_k\rangle$ and $\widehat{R}$ for $|\psi_\pm\rangle$? Why does a joint eigenbasis of $\widehat{O}_n$, $\widehat{R}$, and H exist if and only if $n \in \{1, 2\}$? To answer this question calculate the action of $\widehat{R}$ on the wave vector

$$|\psi_{k,\pm}\rangle = \sum_{l=1}^{n} e^{2\pi\imath lk/n} \widehat{O}_n^l (1 \pm \widehat{R})|\psi\rangle, \quad k \in \{1, \ldots, n\}. \tag{9}$$