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Symmetries in Physics – Exercise Sheet 5

Exercise 5.1 Define the *centraliser* of $a \in G$ to be $C_G(a) = \{g \in G : ga = ag\}$ and the *centre* of G to be $Z(G) = \{h \in G : gh = hg \quad \forall g \in G\}$.

(i) Show that $C_G(a)$ and $Z(G)$ are both subgroups of G and $Z(G)$ is Abelian and normal while $C_G(a)$ is in general not.

(ii) Find a bijection between $G/C_G(a)$ and the conjugacy class (a) of a.

(iii) A group G is called a p-group if it is of order p^n for some prime p and $n \in \mathbb{N}$. Show that a conjugacy class C has size divisible by p unless $C = \{g\}$ with $g \in Z(G)$.

(iv) Let $J \subset G$ contain one and only one element from each conjugacy class of G. Prove the "class equation"

$$|G| = \sum_{x \in J} |G/C_G(x)| = |Z(G)| + \sum_{x \in J \setminus Z(G)} |G/C_G(x)|. \tag{1}$$

(v) Show that the centre of a p-group is non-trivial.

Exercise 5.2

Let us consider the set of Pauli matrices and the two-dimensional unit-matrix,

$$\Sigma = \{\mathbb{1}_2, \sigma_1, \sigma_2, \sigma_3\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \imath \\ -\imath & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{2}$$

(i) Why is the set Σ not a group with respect to matrix multiplication? Extend Σ to a group G by closing the matrix multiplication.

(ii) Calculate the centre of G denoted by $Z(G)$ and identified it with a discrete group you already know.

(iii) Calculate the coset $G/Z(G)$ which is a quotient group. Note that the result will look confusing to you. Interpret and explain this result.

(iv) Calculate all conjugacy classes.

(v) Calculate the centralizer of all elements.

(vi) The group G is a p-group. What is p and what is n? Choose a suitable subset $J \subset G$ and prove Eq. (1) for this particular case.

Exercise 5.3

An application in lattice QCD: Let us define the $2d+1$-dimensional Euclidean Dirac operator

$$\mathcal{D} = \sum_{\mu=1}^{2d+1} \gamma_\mu D_\mu \tag{3}$$

where D_μ are covariant derivatives in the specific directions. The Euclidean γ-matrices fulfill the anti-commutation relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \tag{4}$$

and are $2^d \times 2^d$ matrices. This Dirac operator is put on a lattice where $l \in \mathbb{N}_0$ directions have an even number of lattice sites, say $\mu = 1, \ldots, l$, and $2d+1-l$ directions have an odd number of lattice site (e.g. $d=1$ and $l=1$, then the lattice is given by $\mathbb{Z}_{L_1} \times \mathbb{Z}_{L_2} \times \mathbb{Z}_{L_3}$ with L_1 even and $L_{2/3}$ odd and $\mathbb{Z}_m = \mathbb{Z}/(m\mathbb{Z})$ for a positive integer m). Assuming a direction, say ν, has an even number of lattice sites one can define an operator T_ν assigning to each second lattice site in this direction a minus sign and keeping the other lattice sites as they are. Then one can show that the commutation relation of the operator T_ν with an arbitrary covariant derivative D_μ is

$$T_\nu D_\mu = (-1)^{\delta_\mu \nu} D_\mu T_\nu = \begin{cases} -D_\mu T_\nu, & \mu = \nu, \\ D_\mu T_\nu & \mu \neq \nu. \end{cases} \tag{5}$$

Moreover the T_ν commute under each other as well as with the γ-matrices, i.e.

$$T_\nu T_\mu = T_\mu T_\nu \quad \text{and} \quad T_\nu \gamma_\mu = \gamma_\mu T_\nu. \tag{6}$$

Let us definer the set

$$\Sigma = \{\mathbb{1}, L_1\gamma_1, L_2\gamma_2, \ldots, L_l\gamma_l\}. \tag{7}$$

(i) Show that all elements of Σ, apart from the unit matrix, anti-commute with the Dirac operator.

(ii) Construct a group G out of the set Σ by closing its matrix multiplication.

(iii) Construct a subgroup of G denoted by H whose elements commute with D.

(iv) Calculate the center of the subgroup H denoted by $Z(H)$.

(v) A maximal Abelian subgroup A of a group B is an Abelian subgroup of B such that: If $b \in B$ with $ba = ab$ $\forall a \in A$ then $b \in A$. Calculate a maximal Abelian subgroup of the quotient group $H/Z(H)$ (notice that such a group is not unique). Explain why its order is the degree of degeneracy of the Dirac operator D. What is the order?