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Symmetries in Physics – Exercise Sheet 6

Exercise 6.1

The concept of a group representation on V as a map $\rho : G \to \mathrm{Gl}(V)$ is generalised by the notion of a *group action* on a set Ω. A group action on Ω is defined as $\cdot : G \times \Omega \to \Omega$ such that,

(a) $g \cdot (h \cdot x) = (gh) \cdot x \qquad \forall g, h \in G$ and $x \in \Omega$

(b) $e \cdot x = x$.

where e is the identity element.

(i) Show that a group representation on V defines a group action on V.

(ii) For an arbitrary group action, show that $\rho_g : \Omega \to \Omega$ defined by $\rho_g(x) = g \cdot x$ is a bijection and hence show that $g \mapsto \rho_g$ is a homomorphism from G to $\mathrm{Sym}(\Omega)$, where $\mathrm{Sym}(\Omega)$ is the group of permutations of the elements of Ω.

(iii) Let G be a group with subgroup H such that $|G/H| = n$. By considering an action of G, namely $g \times \widetilde{g}H \to T_g(\widetilde{g})H$, on G/H show that G possess a normal subgroup N such that $|G/N| \leq n!$. Advice: Construct a group N by considering the kernel of the representation of T, i.e. $g \in G$ is in the kernel if $T_g(\widetilde{g})H = T_e(\widetilde{g})H = \widetilde{g}H$.

Exercise 6.2

Let us consider the $m \times m$ matrix

$$T = \begin{pmatrix} 0 & 1 & 0 & & \cdots & 0 \\ & 0 & 1 & 0 & & \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ & & & & & \\ 0 & & & & 0 & 1 \\ 1 & 0 & & \cdots & & 0 \end{pmatrix}, \tag{1}$$

i.e. $T_{ij} = 1$ for all $j - i = 1$ and for $i = m$ and $j = 1$ and otherwise $T_{ij} = 0$.

(i) Construct a group, $\widetilde{G}$, from T via matrix multiplication. The matrix group $\widetilde{G}$ is a representation of a finite group, G, you already know. Which group is this?

(ii) With help of Maschke's theorem we know that the representation $\widetilde{G}$ is completely reducible. What are all irreducible representations of $\widetilde{G}$? Calculate for this purpose the eigenspaces of T and show that they are submodules of the representation $\widetilde{G}$.

(iii) What are the characters of all irreducible representations of $\widetilde{G}$? Interpret them quantum mechanically.

Exercise 6.3

An application in molecular physics: Consider a molecule, in the presence of no external forces, confined to a plane with $N > 1$ interacting atoms at coordinates $\mathbf{x}_i^0 + \mathbf{x}_i = (x_i^0 + x_i, y_i^0 + y_i)$, $i = \{1, \ldots, N\}$ where (x_i^0, y_i^0) is the equilibrium position of the atom i which has a mass m_i.

(i) Explain why the Hamiltonian may be written as

$$H = \frac{1}{2}\dot{\mathbf{x}}^T \mathbf{M^2} \dot{\mathbf{x}} + \mathbf{x^T V x}, \tag{2}$$

$\mathbf{x} = (x_1, y_1, x_2, y_2, \ldots)$, for small oscillations of the atoms about their equilibrium positions. What is the significance of the matrices $\mathbf{M}$ and $\mathbf{V}$?

(ii) Show that if the molecule is invariant under the transformations $\mathbf{x} = \mathbf{Q}(g)\mathbf{x}'$, where $\mathbf{Q}$ forms a representation of a group G, then this implies the commutator $[\mathbf{V}', \mathbf{Q}']_- = \mathbf{0}$ where $\mathbf{Q}' = \mathbf{MQM^{-1}}$ and $\mathbf{V}' = \mathbf{M^{-1}VM^{-1}}$.

(iii) A normal mode of the system is defined as a solution to the equations of motion in which all atoms oscillate with the same frequency. Show that for a normal mode, $\mathbf{Mx}$ is an eigenvector of $\mathbf{V}'$ and that if $\mathbf{Mx}$ is an eigenvector so is $\mathbf{Q}'\mathbf{Mx}$. Under what conditions is $\mathbf{Q}'$ a reducible representation and if so what do the invariant subspaces correspond to?

(iv) Let us consider two examples of the system with identical particles ($m_i = m_j$ for all $i, j = 1, 2, \ldots$), e.g. a benzol ring. The group shall be in both cases $G = C_N$, only the matrix representation shall be different. In the first case the generator of the representation, T_1, rotates each particle with the same angle, i.e.

$$\mathbf{x}_i \rightarrow \begin{pmatrix} \cos\left(\frac{2\pi}{N}\right) & \sin\left(\frac{2\pi}{N}\right) \\ -\sin\left(\frac{2\pi}{N}\right) & \cos\left(\frac{2\pi}{N}\right) \end{pmatrix} \mathbf{x}_i. \tag{3}$$

In the second case, the generator of the representation, T_2, performs a cyclic permutation of the particles, $\mathbf{x}_i \rightarrow \mathbf{x}_{i+1}$ and $\mathbf{x}_N \rightarrow \mathbf{x}_1$. What is the relation of these two representations? What does each of the two symmetries tell us about the matrix V? Into which irreducible representations split the representations generated by T_1 and T_2?