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Symmetries in Physics – Exercise Sheet 7

Exercise 7.1

Let $D(g)$ be a representation of a group G and $D^*(g)$ be the complex conjugate of $D(g)$.

(i) Show that $D^*(g)$ is also a representation.

(ii) Let us assume that D and D^* are equivalent representations, i.e. $D^*(g) = C^{-1}D(g)C$. Proof that if D is irreducible then $CC^* = \lambda\mathbb{I}$ with $\lambda \in \mathbb{C}$.

(iii) If D is unitary show also that $CC^\dagger = \mu\mathbb{I}$ with $\mu \in \mathbb{C}$. Moreover show that C may be rescaled to be either symmetric or anti-symmetric.

Exercise 7.2

Let G be a finite group. The characters χ are traces of a representation D. Let us label the irreducible representations D^μ by μ and the corresponding characters by $\chi^\mu(g) = \mathrm{tr} D^\mu(g)$ for $g \in G$. The dimension of this irreducible representation is $n^\mu = \chi^\mu(e)$ where e is the identity element of G. The order of G is denoted by $|G|$.

(i) Show that

$$\sum_{g\in G} \chi^\mu(g)\chi^\nu(hg^{-1}) = \frac{|G|}{n_\mu}\chi^\mu(h)\delta_{\mu\nu} \tag{1}$$

for all $h \in G$ and representations μ, ν.

(ii) For an arbitrary representation D of G on a space V define the operator

$$P^\mu = \frac{n_\mu}{|G|}\sum_{g\in G} \chi^\mu(g)D(g), \tag{2}$$

where μ labels an irreducible representation of G. Show that these operators are orthogonal projections, i.e.

$$P^\mu P^\nu = P^\mu \delta_{\mu\nu}, \tag{3}$$

(iii) What effect does P^μ have on an arbitrary vector in V?

Exercise 7.3

An application in quantum mechanics:
Let us consider a Hamilton operator

$$H\psi(x,y) = \frac{\psi(x+1,y)+\psi(x,y+1)-4\psi(x,y)+\psi(x-1,y)+\psi(x,y-1)}{4} + V(x,y)\psi(x,y). \quad (4)$$

on a lattice $\mathbb{L}$ of a two-dimensional torus whose periodicity is $x + L_x = x$ and $y + L_y = y$ with some $L_x, L_y \in \mathbb{N}$. The first term of H is the discrete version of the Laplacian.

(i) Explain why we need that the potential has to be periodic $V(x+L_x, y) = V(x, y+L_y) = V(x,y)$ to have a well-defined system.

(ii) Explain why we can generate any lattice site via

$$r(g_x, g_y) := D(g_x \times g_y)r_0 = (D_x(g_x)x_0, D_y(g_y)y_0) \in \mathbb{L} \quad (5)$$

with $r_0 = (x_0, y_0) \in \mathbb{L}$ a fixed site of the lattice and D a representation of the product group $C_{L_x} \times C_{L_y}$. What is the concrete group action of $C_{L_x} \times C_{L_y}$ on the lattice? Recall that we can represent C_n by taking powers of a generator T.

(iii) Let us define a corresponding operator representation acting on the wavefunction ψ via

$$\widehat{D}(g_x \times g_y)\psi(r) := \psi(D(g_x \times g_y)r). \quad (6)$$

This yields us again a representation of the product group $C_{L_x} \times C_{L_y}$ on an $L_x L_y$ dimensional vectorspace since we do not assume any inner symmetries of the system. Let the basis of this vector space be given by $\{\psi(r)\}_{r\in\mathbb{L}}$. Completely decompose the representation $\widehat{D}$ to its irreducible representations. Build a new basis $\{\phi^\mu\}$ out of the irreducible representations and find a bijection of this set to the characters of the irreducible representations of $C_{L_x} \times C_{L_y}$.

(iv) Perform the change of basis $\{\psi(r)\}_{r\in\mathbb{L}} \to \{\phi^\mu\}$ in the Schrödinger equation (4). To what transformation does this change correspond to. Interpret the characters of the irreducible representations in this picture.