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Symmetries in Physics – Exercise Sheet 8

Exercise 8.1

The lemma of Schur can be inverted for a unitary representation D of a group G to find out if D is irreducible. Thus prove the following statement:

Let D be a unitary representation of the group G and the following holds: $D(g)B = BD(g)\ \forall g \in G \ \Leftrightarrow\ B = \mu\mathbb{1},\ \mu \in \mathbb{C}$. Then D is irreducible.

Notice that this statement is not only true for discrete and compact groups. It applies to non-compact groups, too. The reason is that unitary representations are always compact, also for non-compact groups.

Exercise 8.2

Let us find all irreducible representations of the dihedral group D_n. Recall that each element in D_n can be written as a product of a reflection R ($R^2 = e$) and a rotation T ($T^n = e$) with the "commutation relation" $RT = T^{-1}R$, i.e. $D_n = \{T^j, RT^j | j = 0, \ldots, n-1\}$.

(i) Calculate all conjugacy classes of D_n. Why do we have to distinguish between even and odd n?

(ii) Show that the map

$$D^\mu(R) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ D^\mu(T) = \begin{pmatrix} \cos\left(\frac{2\pi\mu}{n}\right) & \sin\left(\frac{2\pi\mu}{n}\right) \\ -\sin\left(\frac{2\pi\mu}{n}\right) & \cos\left(\frac{2\pi\mu}{n}\right) \end{pmatrix} \tag{1}$$

generates an irreducible representation for $\mu \in \{0, \ldots, n-1\} \setminus \{0\}$ if n is odd and $\mu \in \{0, \ldots, n-1\} \setminus \{0, n/2\}$ if n is even. Apply for this calculation the lemma proven in Exercise 8.1. Why do we have to exclude $\mu = 0$ for all n and $\mu = n/2$ for n even? What are the corresponding irreducible representations of the exclusive cases and what dimension do they have?

(iii) Why are the irreducible representations (1) all irreducible representations of D_n? Recall for this the relation between the dimension of the irreducible representations and the order of the discrete group. Relate the representations (1) to the conjugacy classes of D_n and calculate the character table of D_n.

(iv) Show that the map to $n \times n$ dimensional matrices,

$$\hat{D}(R) = \begin{pmatrix} & & & 1 \\ & 0 & 1 & \\ & & & \\ & 1 & 0 & \\ 1 & & & \end{pmatrix},\ \hat{D}(T) = \begin{pmatrix} 0 & 1 & 0 & & \cdots & 0 \\ & 0 & 1 & 0 & & \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & 0 & 1 \\ 1 & 0 & & \cdots & & 0 \end{pmatrix}, \tag{2}$$

is a representation of D_n and decompose it in all its irreducible representations. Advice: Employ that irreducible characters are orthogonal and, thus, build a vector space (see lecture notes) and the construction of the projection operator in Exercise 7.2.

Exercise 8.3

An application in many particle systems:

Let us consider a system of N coupled identical spin-$1/2$ particles in a magnetic field B with the Hamilton operator

$$H = \sum_{1\leq a,b\leq N} J_{ab}(\sigma_x^{(a)}\sigma_x^{(b)} + \sigma_y^{(a)}\sigma_y^{(b)} + \sigma_z^{(a)}\sigma_z^{(b)}) + \sum_{1\leq a\leq N} (B_x\sigma_x^{(a)} + B_y\sigma_y^{(a)} + B_z\sigma_z^{(a)}), \qquad (3)$$

where $\sigma_\mu^{(j)}$ is the μ'th Pauli matrix to the j'th particle. Notice that the Pauli matrices for different particles lie in different tensor spaces. The coefficients $J_{ab} = J_{ba}$ are coupling constants. The whole system is $\mathrm{SU}(2)$ invariant if $B = 0$, i.e. invariant under $\sigma_\mu^{(j)} \to U\sigma_\mu^{(j)}U^\dagger$ for all $j = 1,\ldots,N$ for $U \in \mathrm{SU}(2)$. Let us assume that at finite magnetic field $B \neq 0$ the system is invariant under C_m embedded in the following way

$$D(T^j) = \begin{pmatrix} e^{2\pi\imath j/m} & 0 \\ 0 & e^{-2\pi\imath j/m} \end{pmatrix} \in \mathrm{SU}(2) \qquad (4)$$

such that the Hamiltonian is invariant under

$$[D(T^j)]^{\otimes N} H [D(T^{-j})]^{\otimes N} = H \qquad (5)$$

for all $j = 0,\ldots,m-1$.The notation $[D(T^j)]^{\otimes N}$ abbreviates $[D(T^j)]^{\otimes N} = D(T^j)\otimes D(T^j)\otimes\cdots\otimes D(T^j)$ where $D(T^j)$ appears N-times and acts in a natural way on the product space of spins.

(i) First calculate $D(T^j)\sigma_\mu^{(a)}D(T^{-j})$. This yields a three dimensional representation $\widehat{D}$ acting as a rotation group on $\mathbb{R}^3$. Write up explicitly what this map, $T^j \in C_m \to \widehat{D}(T^j) \in \mathrm{SO}(3)$, is.

(ii) Show that the first sum in Eq. (5) is invariant under this transformation. What is the invariance condition on B for this transformation?

(iii) We know that all irreducible representations of C_m are one-dimensional. Decompose the representations D and $\widehat{D}$ in the one-particle space as well as $[D(T^j)]^{\otimes N}$ in the N-particle space. Interpret your results from a physics point of view. What does this decomposition in irreducible representations mean for the Hamiltonian?