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Symmetries in Physics – Exercise Sheet 9

Exercise 9.1

The lemma of Schur can be modified such that one can apply it to the Lie-algebra of Lie-groups. Thus prove the following statement:

Let D be a unitary representation of a Lie-group G and $\widehat{D}$ the corresponding representation of the Lie-algebra $\mathrm{Lie}(G)$. Then the following holds: $\widehat{D}(g)B = B\widehat{D}(g)\ \forall g \in \mathrm{Lie}(G) \ \Leftrightarrow\ B = \mu\mathbb{1},\ \mu \in \mathbb{C}$. Then $\widehat{D}$ and, hence, D are irreducible.

Notice that this statement holds for compact as well as non-compact groups as it was already the case in Exercise 8.1. The reason is again the unitarity of the representation which is always compact.

Exercise 9.2

Let us consider the group $\mathrm{U}(1)$.

(i) Let $\varphi \in [0, 2\pi[$. Show that the map

$$D^{(m)}(e^{\imath\varphi}) = \begin{pmatrix} \cos(m\varphi) & \sin(m\varphi) \\ -\sin(m\varphi) & \cos(m\varphi) \end{pmatrix} \in \mathrm{SO}(2) \tag{1}$$

is a representation of $\mathrm{U}(1)$ for certain m. From which number set has the variable m to be drawn? Identify m with a quantity in physics.

(ii) What is the map of the Lie-algebra of $\mathrm{U}(1)$ to the representation $\widehat{D}^{(m)}$ corresponding to the representation $D^{(m)}$? Check the reducibility of the representation $D^{(m)}$ via the lemma you have proven in Exercise 9.1. What are the irreducible representations encoded in the representation $D^{(m)}$?

Exercise 9.3

Refreshing knowledge about quantum mechanics:

Let us consider the Lie-algebra

$$[T_a, T_b]_- = \imath\epsilon^{abc}T_c, \quad a, b \in \{1, 2, 3\} \tag{2}$$

where ϵ^{abc} is the Levi-Cevita symbol (totally anti-symmetric in the indices and $\epsilon^{123} = 1$).

(i) Show that the following three maps are representations of the Lie-algebra (2):

$$\widehat{D}_{\mathrm{fun}}(T_1, T_2, T_3) = \left(\frac{1}{2}\sigma_x, \frac{1}{2}\sigma_y, \frac{1}{2}\sigma_z\right), \quad (\sigma_\mu \text{ are the Pauli matrices}), \tag{3}$$

$$\widehat{D}_{\mathrm{adj}}(T_1, T_2, T_3) = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \imath \\ 0 & -\imath & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \imath \\ 0 & 0 & 0 \\ -\imath & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \imath & 0 \\ -\imath & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right), \tag{4}$$

$$\widehat{D}_{\mathrm{uni}}(T_1, T_2, T_3) = \left(\imath\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \imath\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \imath\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\right). \tag{5}$$

What significance do they play in physics? Which classical groups $G_{\rm fun/adj}$ do the generators (3) and (4) represent?

(ii) Show with help of Exercise 9.1 that the representations (3) and (4) yield irreducible group representations while the representation (5) does not. In the last case consider the algebra of quadratic forms in the coordinates and differentials, i.e. $\mathcal{A} = \{x_a x_b, x_a\partial_{x_b}, \partial_{x_a}\partial_{x_b}\}$. The algebra $\mathcal{A}$ is a closed algebra of the operator algebra on a Hilbert space (functions depending on coordinates and moments, e.g. the Hamilton operator).

(iii) Let us employ the Euler angles as coordinates of the corresponding groups

$$\begin{aligned} U_{\rm fun} &= \exp[\imath\varphi_1^{\rm (fun)}\widehat{D}_{\rm fun}(T_2)]\exp[\imath\varphi_2^{\rm (fun)}\widehat{D}_{\rm fun}(T_3)]\exp[-\imath\varphi_3^{\rm (fun)}\widehat{D}_{\rm fun}(T_2)], &(6)\\ U_{\rm adj} &= \exp[\imath\varphi_1^{\rm (adj)}\widehat{D}_{\rm adj}(T_2)]\exp[\imath\varphi_2^{\rm (adj)}\widehat{D}_{\rm adj}(T_3)]\exp[-\imath\varphi_3^{\rm (adj)}\widehat{D}_{\rm adj}(T_2)], &(7)\\ U_{\rm uni} &= \exp[\imath\varphi_1^{\rm (uni)}\widehat{D}_{\rm uni}(T_2)]\exp[\imath\varphi_2^{\rm (uni)}\widehat{D}_{\rm uni}(T_3)]\exp[-\imath\varphi_3^{\rm (uni)}\widehat{D}_{\rm uni}(T_2)]. &(8)\end{aligned}$$

Then show that

$$\begin{pmatrix} U_{\rm fun}\sigma_x^{(a)}U_{\rm fun}^{-1} \\ U_{\rm fun}\sigma_y^{(a)}U_{\rm fun}^{-1} \\ U_{\rm fun}\sigma_z^{(a)}U_{\rm fun}^{-1} \end{pmatrix} = U_{\rm adj}\begin{pmatrix} \sigma_x^{(a)} \\ \sigma_y^{(a)} \\ \sigma_z^{(a)} \end{pmatrix} \qquad (9)$$

and

$$U_{\rm uni}f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = f\left(U_{\rm adj}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right), \qquad (10)$$

where f is a smooth function. What is the relation between the different angles? Explain why the relation (10) is a bijection between two different representation of one and the same group while Eq. (9) is not a bijection. Interpret your results in a physics context.

(iv) The representation (1) of Exercise 9.2 can be embedded in the representation (7) via

$$\begin{pmatrix} \cos(m\varphi) & \sin(m\varphi) \\ -\sin(m\varphi) & \cos(m\varphi) \end{pmatrix} \rightarrow \begin{pmatrix} \cos(m\varphi) & \sin(m\varphi) & 0 \\ -\sin(m\varphi) & \cos(m\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (11)$$

What is the corresponding embedding in the representations (6) and (8)? Discuss in this context that the representation (1) is reducible although the representation (7) is irreducible.