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Symmetries in Physics – Exercise Sheet 10

Exercise 10.1

A spin-angular momentum coupling

We consider the action of a non-relativistic, spin $1/2$ particle of mass m in an electro magnetic vectorfield $A_\mu(x)$ and with an external electrical current $j_\mu(x)$,

$$S = \int d^4x \left((D_+\psi)^\dagger D_-\psi + m^2\psi^\dagger\psi + j_0(x)\psi^\dagger\psi - \sum_{a=1}^{3} j_a(x)\psi^\dagger\sigma_a\psi \right), \tag{1}$$

with the covariant derivatives

$$D_\pm = \mathbb{1}_2 D_0 \pm \sum_{a=1}^{3} \sigma_a D_a, \quad D_\mu = \imath\partial_\mu - A_\mu(x). \tag{2}$$

Note that $\psi = (\psi_1, \psi_2)$ is a two component vector valued function. The 2×2 matrices σ_a are the three Pauli matrices.

(i) In Exercise 9.3 you learnt the relation between the two compact Lie groups $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$. Let $A_\mu = j_\mu = 0$. Show that the covariant derivatives are invariant under the global transformation

$$D_\pm \to U^\dagger D_\pm U, \quad \mathbf{r} \to O\mathbf{r} \tag{3}$$

for suitable $O \in \mathrm{SO}(3)$ and $U \in \mathrm{SU}(2)$ ($\mathbf{r}$ is the three dimensional spatial vector). What is the exact relation between O and U?

(ii) Due to Noether's theorem there is a conserved charge with respect to the transformation (3). Calculate this charge and what is its physical significance?

(iii) What is the equation of motion of this charge when switching on the external sources A_μ and j_μ?

Exercise 10.2

The Coulomb potential

Let us consider the Hamiltonian of a particle in a Coulomb potential

$$H = \left(\frac{||\mathbf{p}||^2}{2m} - \frac{V}{||\mathbf{r}||} \right), \tag{4}$$

where m is the mass of the particle and V some coupling constant. We denote the norm of a vector as $||.||$ and the scalar product as $\langle .,. \rangle$.

(i) What are the corresponding equations of motion?

(ii) Consider the two kinds of infinitesimal transformations

$$\mathbf{r} \to T_1(\mathbf{r}) = \mathbf{r} + \delta O\mathbf{r} \tag{5}$$

with an infinitesimal anti-symmetric 3×3 matrix $\delta O = -\delta O^T$ and

$$\mathbf{r} \to T_2(\mathbf{r}) = \mathbf{r} + 2\langle \mathbf{v}, \mathbf{r} \rangle \mathbf{p} - \langle \mathbf{v}, \mathbf{p} \rangle \mathbf{r} - \langle \mathbf{r}, \mathbf{p} \rangle \mathbf{v} \tag{6}$$

with an infinitesimal three-dimensional vector $\mathbf{v}$. Show that both infinitesimal transformations correspond to symmetries of the Hamiltonian (4). For this purpose calculate the transformations of the momentum $\mathbf{p}$ since the equations of motion have to be invariant. Then expand the Hamiltonian in this representation to the first order of δO and $\mathbf{v}$.

(iii) Noether's theorem tells us that to both symmetries exist conserved quantities. Show that these quantities are

$$\begin{aligned} \mathbf{L} &= \mathbf{r} \times \mathbf{p}, & (7) \\ \mathbf{A} &= \mathbf{p} \times (\mathbf{r} \times \mathbf{p}) - mV\mathbf{r}/||\mathbf{r}||, & (8) \end{aligned}$$

where "$\times$" is the exterior product. Explain their physical significance. Recall that we have also an invariance under time translation such that the energy (4) is also a constant.

(iv) Let us rescaled the vector $\mathbf{A}$ to

$$\widetilde{\mathbf{A}} = \frac{\mathbf{p} \times (\mathbf{r} \times \mathbf{p}) - mV\mathbf{r}/||\mathbf{r}||}{\sqrt{||\mathbf{p}||^2 - 2mV/||\mathbf{r}||}}. \tag{9}$$

The Poisson bracket is defined by

$$\{f, g\} = \sum_{a=1}^{3} \left(\frac{\partial f}{\partial r_a} \frac{\partial g}{\partial p_a} - \frac{\partial g}{\partial r_a} \frac{\partial f}{\partial p_a} \right). \tag{10}$$

Show that the components of the three-dimensional vectors $\mathbf{L}$ and $\widetilde{\mathbf{A}}$ fulfill the same algebra with the Poisson bracket as the Lie algebra of $\mathrm{SO}(4)$ with the commutator, in particular prove the following relations

$$\begin{aligned} \{\langle \mathbf{v}, \mathbf{L} \rangle, \langle \widetilde{\mathbf{v}}, \mathbf{L} \rangle\} &= \langle \mathbf{v} \times \widetilde{\mathbf{v}}, \mathbf{L} \rangle, \\ \{\langle \mathbf{v}, \mathbf{L} \rangle, \langle \widetilde{\mathbf{v}}, \widetilde{\mathbf{A}} \rangle\} &= \langle \mathbf{v} \times \widetilde{\mathbf{v}}, \widetilde{\mathbf{A}} \rangle, \\ \{\langle \mathbf{v}, \widetilde{\mathbf{A}} \rangle, \langle \widetilde{\mathbf{v}}, \widetilde{\mathbf{A}} \rangle\} &= -\langle \mathbf{v} \times \widetilde{\mathbf{v}}, \mathbf{L} \rangle \end{aligned} \tag{11}$$

with $\mathbf{v}$ and $\widetilde{\mathbf{v}}$ two arbitrary fixed three-dimensional vectors. Choose a suitable orthonormal basis of the Lie algebra of $\mathrm{SO}(4)$ and calculate and compare its structure constants with the Poisson brackets (11).

What does this result tell us about the quantum mechanical system corresponding to the Hamiltonian (4)?

We wish you a wonderful christmas and a happy new year!