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Symmetries in Physics – Exercise Sheet 11

Exercise 11.1

Let G be a simply connected, compact Lie-group of dimension d (it has d linearly independent generators and each element $g \in G$ can be generated by $g = \exp[\imath T]$ with $T \in \mathrm{Lie}\, G$) and $D_G^{(\mathrm{adj})}$ be the adjoint representation of G.

(i) Prove that $D_G^{(\mathrm{adj})}$ is a subgroup of $D_{\mathrm{SO}(d)}^{(\mathrm{fund})}$ which is the fundamental representation of the special orthogonal group $\mathrm{SO}(d)$.

(ii) Let $\{T_1, \ldots, T_d\}$ be an orthonormal basis of the Lie-algebra, $\mathrm{Lie}\, D_G^{(\mathrm{fund})}$, of G in the fundamental representation, i.e. $\mathrm{tr}(T_a T_b) = \delta_{ab}$, and $D_G^{(\mathrm{fund})}$ its fundamental representation. Show that the adjoint representation of the group G is equal to

$$\begin{array}{ccc} D_G^{(\mathrm{fund})} \times \mathrm{Lie}\, D_G^{(\mathrm{fund})} & \longrightarrow & \mathrm{Lie}\, D_G^{(\mathrm{fund})} \\ A \times \sum_{c=1}^{d} \alpha_c T_c & \longmapsto & \sum_{c=1}^{d} \alpha_c A T_c A^{-1} \end{array} \quad (1)$$

with some real coefficients $\alpha_c \in \mathbb{R}$. In particular show that the map

$$\begin{array}{ccc} D_G^{(\mathrm{fund})} & \longrightarrow & D_G^{(\mathrm{adj})} \\ A & \longmapsto & B = \{\mathrm{tr}(T_a A T_b A^{-1})\}_{1 \leq a,b \leq d} \end{array} \quad (2)$$

gives a coordinate representation for $D_G^{(\mathrm{adj})}$, meaning that it is surjective and is exactly the adjoint representation. To prove this show that the structure constants generate the algebra of the representation (2).

Exercise 11.2

Let us consider the three groups $\mathrm{SO}(n) \subset \mathrm{U}(n)$ (special orthogonal group), $\mathrm{SU}(n) \subset \mathrm{U}(n)$ (special unitary group), and $\mathrm{USp}(2n) \subset \mathrm{U}(2n)$ (unitary symplectic group) denoted by the Dyson index $\beta = 1, 2, 4$, respectively. All three groups are defined in their fundamental representations by

$$\begin{cases} UU^T = \mathbb{1}_n, & \beta = 1, \\ UU^\dagger = \mathbb{1}_n, & \beta = 2, \\ U \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} U^T = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}, & \beta = 4, \end{cases} \quad (3)$$

and the uni modularity

$$\det U = 1. \quad (4)$$

Recall that $(.)^T$ and $(.)^\dagger$ are the transpose and the adjoint of a matrix, respectively.

(i) Show that the corresponding Lie algebras, $U = \exp[\imath A]$, in the fundamental representations are given by

$$\left\{\begin{array}{cc} A = A^\dagger = -A^T, & \beta = 1, \\ A = A^\dagger, & \beta = 2, \\ A = A^\dagger = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} A^T \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}, & \beta = 4, \end{array}\right. \tag{5}$$

and the traceless condition

$$\mathrm{tr} A = 1. \tag{6}$$

What is the dimension d of the three kinds of Lie-groups in terms of the dimension N of the fundamental representation of the Lie groups?

(ii) Let $\mathbf{e}_j$ be the normalized n-dimensional vector with a 1 at the j'th position and otherwise zero and $\{\sigma_1, \sigma_2, \sigma_3\}$ the three Pauli matrices. Then show that the matrix sets

$$\bigcup_{1 \leq a < b \leq n} \left\{ \frac{\mathbf{e}_a \otimes \mathbf{e}_b - \mathbf{e}_b \otimes \mathbf{e}_a}{\sqrt{2}\imath} \right\}, \tag{7}$$

for $\beta = 1$,

$$\left[\bigcup_{1 \leq a \leq n-1} \left\{ \frac{1}{\sqrt{a(a+1)}} \left(\sum_{j=1}^{a} \mathbf{e}_j \otimes \mathbf{e}_j - a\, \mathbf{e}_{a+1} \otimes \mathbf{e}_{a+1} \right) \right\} \right] \cup \left[\bigcup_{1 \leq a < b \leq n} \left\{ \frac{\mathbf{e}_a \otimes \mathbf{e}_b - \mathbf{e}_b \otimes \mathbf{e}_a}{\sqrt{2}\imath}, \frac{\mathbf{e}_a \otimes \mathbf{e}_b + \mathbf{e}_b \otimes \mathbf{e}_a}{\sqrt{2}} \right\} \right] \tag{8}$$

for $\beta = 2$, and

$$\left[\bigcup_{1 \leq a \leq n} \left\{ \frac{\mathbf{e}_a \otimes \mathbf{e}_a \otimes \sigma_1}{\sqrt{2}}, \frac{\mathbf{e}_a \otimes \mathbf{e}_a \otimes \sigma_2}{\sqrt{2}}, \frac{\mathbf{e}_a \otimes \mathbf{e}_a \otimes \sigma_3}{\sqrt{2}} \right\} \right] \cup \left[\bigcup_{1 \leq a < b \leq n} \left\{ \frac{(\mathbf{e}_a \otimes \mathbf{e}_b - \mathbf{e}_b \otimes \mathbf{e}_a) \otimes \mathbb{1}_2}{2\imath}, \frac{(\mathbf{e}_a \otimes \mathbf{e}_b + \mathbf{e}_b \otimes \mathbf{e}_a) \otimes \sigma_3}{2} \right\} \right] \cup \left[\bigcup_{1 \leq a < b \leq n} \left\{ \frac{(\mathbf{e}_a \otimes \mathbf{e}_b + \mathbf{e}_b \otimes \mathbf{e}_a) \otimes \sigma_1}{2}, \frac{(\mathbf{e}_a \otimes \mathbf{e}_b + \mathbf{e}_b \otimes \mathbf{e}_a) \otimes \sigma_2}{2} \right\} \right] \tag{9}$$

for $\beta = 4$ are orthonormal bases of the corresponding Lie algebras.

(iii) Calculate the completeness relation for the three Lie algebras with help of the given bases, in particular calculate $(\sum_{j=1}^{d} T_{ab}^{j} T_{cd}^{j})/d$ where T^j are the generators (7), (8), and (9). For this purpose it is simpler to calculate the sum $[\sum_{j=1}^{d} \mathrm{tr}(M_1 T^j)\mathrm{tr}(M_2 T^j)]/d$ for two arbitrary matrices M_1 and M_2. Why do we not obtain the unit matrix, only? Interpret the results via the symmetries of the elements of the Lie algebra.

(iv) Calculate the structure constants of the three Lie-Algebras.

(v) Show that all three groups are semi-simple. For this purpose calculate the Cartan metric. Why is $\mathrm{U}(n)$ not semi-simple? For what dimension n are the three groups also simple?