Universität Bielefeld

G. Akemann, M. Kieburg

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Symmetries in Physics – Exercise Sheet 12

Exercise 12.1

Recall that the Cartan-subalgebra $\mathcal{C}$ is an Abelian algebra of a Lie-algebra $\mathcal{L}$ fulfilling the condition: $Y \in \mathcal{L}$ with $[Y, X]_- = 0\ \forall Y \in \mathcal{C} \Rightarrow X \in \mathcal{C}$. Hence the Cartan-subalgebra and, thus, the Cartan-subgroup are not unique at all. However for a simply connected, compact group G different Cartan-subalgebras, $\mathcal{C}_1$ and $\mathcal{C}_2$, are similar, in particular there is a fixed group element $g \in G$ such that $g\mathcal{C}_1 g^{-1} = \mathcal{C}_2$.

(i) Calculate the Cartan-subalgebras and their corresponding Cartan-subgroups of the Lie-groups $\mathrm{U}(N)$, $\mathrm{SO}(N)$, $\mathrm{SU}(N)$, and $\mathrm{USp}(2N)$. For this purpose show that one can find Cartan-sybalgebras of $\mathrm{U}(N)$, $\mathrm{SU}(N)$, and $\mathrm{USp}(2N)$ as subsets of the diagonal matrices and for $\mathrm{SO}(N)$ as subsets of 2×2 block-diagonal matrices.

(ii) Let us consider the non-compact group $\mathrm{U}(1,1) = \{A \in \mathrm{Gl}(2) | A\,\mathrm{diag}(1,-1)A^\dagger = \mathrm{diag}(1,-1)\}$. Show that there are different Cartan-subalgebras and Cartan-subgroups which are not similar to each other. There are all in all three Cartan-subalgebras. Enlist them all.

(iii) The Weyl-reflection group is another important subgroup of a Lie-group. Let Cart_G be the Cartan subgroup of the Lie-Group G. Then the Weyl-reflection group is $\mathrm{Weyl}_G = \{g \in G | g\mathrm{Cart}_G g^{-1} = \mathrm{Cart}_G\}$. Calculate this group for $\mathrm{U}(N)$, $\mathrm{SO}(N)$, $\mathrm{SU}(N)$, and $\mathrm{USp}(2N)$.
Hint: You have to distinguish the groups $\mathrm{SO}(2N)$ and $\mathrm{SO}(2N+1)$ since you have always to check if g is in the considered group.

Exercise 12.2

Let Δ be the set of roots. Given $\alpha \in \Delta$, define σ_α to be the linear transformation implementing reflection in the hyperplane orthogonal to α.

(i) Show that for a root β this projection explicitly reads

$$\sigma_\alpha(\beta) = \beta - \frac{2\alpha \cdot \beta}{|\alpha|^2}\alpha. \tag{1}$$

(ii) Show that $\sigma_\alpha(\beta) \in \Delta$ with help of the Cartan-Weyl basis introduced in the lecture and of the antisymmetry of any generator in the adjoint representation.
Background: The reflections σ_α are induced by the Weyl-reflections introduced in Exercise 12.1(iii) and are, therefore, the Weyl-reflections in the root space. This can be seen by choosing an orthonormal basis in the Cartan-subalgebra $H = (H_1, \ldots, H_r)$ and the remaining basis vectors of the Lie-algebra $\{E_\alpha\}_{\alpha\in\Delta}$. Let $H_\alpha = 2\alpha \cdot H/|\alpha|^2$ with $\alpha \in \Delta$. Then a Weyl reflection in the Cartan-subalgebra $H \to WH$ is a linear operation (W is an orthogonal $r \times r$ matrix which is the adjoint representation of $g \in \mathrm{Weyl}_G$ acting only on Cart_G) and carries over to one on the root space by $\alpha \cdot (WH) = (W^T\alpha) \cdot H$.

(iii) For two distinct roots, α and β, note that $\alpha \cdot \beta \neq 0$ and the quantization of the roots implies $2\alpha \cdot \beta/|\alpha|^2 = \pm 1$ or $2\alpha \cdot \beta/|\beta|^2 = \pm 1$. Show with help of the quantization of the roots and the

Weyl-reflections that

$$\alpha \cdot \beta < 0 \Rightarrow \alpha + \beta \in \Delta,$$
$$\alpha \cdot \beta > 0 \Rightarrow \alpha - \beta \in \Delta. \tag{2}$$

Exercise 12.3

In this exercise we construct all Casimir operators of a representation of a Lie-algebra $\mathrm{Lie}\, D_G^{(\mathrm{o})}$ for a d-dimensional Lie-group G. For this purpose we denote the fundamental representation of G by $D_G^{(\mathrm{fund})}$ and the corresponding orthonormal basis ($\mathrm{tr}\, T_a^{(\mathrm{f})} T_b^{(\mathrm{f})} = \delta_{ab}$) of the Lie-algebra by $\mathrm{span}\{T_1^{(\mathrm{f})}, \ldots, T_d^{(\mathrm{f})}\} = \mathrm{Lie}\, D_G^{(\mathrm{fund})}$. The basis of the representation $\mathrm{Lie}\, D_G^{(\mathrm{o})}$, namely $\{T_1^{(\mathrm{o})}, \ldots, T_d^{(\mathrm{o})}\}$, is given by the mapping $T_j^{(\mathrm{o})} = \phi(T_j^{(\mathrm{f})})$ of the homomorphism $\phi : D_G^{(\mathrm{fund})} \to D_G^{(\mathrm{o})}$, i.e. $[T_a^{(\mathrm{f})}, T_b^{(\mathrm{f})}]_- = \imath \sum_{c=1}^d f_{abc} T_c^{(\mathrm{f})} \Rightarrow [T_a^{(\mathrm{o})}, T_b^{(\mathrm{o})}]_- = \imath \sum_{c=1}^d f_{abc} T_c^{(\mathrm{o})}$. Recall that an operator A in the operator algebra (this can be matrices but also the algebra of differential operators as in quantum mechanics) closing the representation $\mathrm{Lie}\, D_G^{(\mathrm{o})}$ under matrix multiplication (it is up to now only closed under the Lie-bracket and addition) is a Casimir operator if and only if $[A, T_j^{(\mathrm{o})}]_- = 0$ for all $j = 1, \ldots, d$. We will construct such operators via a larger operator

$$C = \sum_{j=1}^{d} T_j^{(\mathrm{o})} \otimes T_j^{(\mathrm{f})} \tag{3}$$

which lives in the tensor space $\mathrm{Lie}\, D_G^{(\mathrm{o})} \otimes \mathrm{Lie}\, D_G^{(\mathrm{f})}$. Recall that a product of two operators $A_1 \otimes B_1$ and $A_2 \otimes B_2$ is given by $(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2$.

(i) Show that for any $m \in \mathbb{N}$ the operator

$$C_m^{(\mathrm{o})} = \mathrm{tr}_{\,\mathrm{fund}} C^m = \sum_{j_1, \ldots, j_m = 1}^{d} T_{j_1}^{(\mathrm{o})} \cdots T_{j_m}^{(\mathrm{o})} \mathrm{tr}\,(T_{j_1}^{(\mathrm{f})} \cdots T_{j_m}^{(\mathrm{f})}) \tag{4}$$

is a Casimir operator of the representation $\mathrm{Lie}\, D_G^{(\mathrm{o})}$.
Hint: Proof first that $[C, T_b^{(\mathrm{o})}]_- = [T_b^{(\mathrm{f})}, C]_-$ and recall that the commutator of a product of operators is $[A_1 \cdots A_m, B]_- = \sum_{j=1}^m A_1 \cdots A_{j-1} [A_j, B]_- A_{j+1} \cdots A_m$.

(ii) The operators $C_1^{(\mathrm{o})}$ and $C_2^{(\mathrm{o})}$ are of particular interest in physics. What is there significance in physics for the groups $\mathrm{U}(1)$, $\mathrm{SU}(2)$, and $\mathrm{SO}(3)$?
What is $C_1^{(\mathrm{o})}$ and $C_2^{(\mathrm{o})}$ for the Lie-groups $\mathrm{U}(N)$, $\mathrm{SO}(N)$, $\mathrm{SU}(N)$, and $\mathrm{USp}(2N)$? Calculate them explicitly for the fundamental and the adjoint representation. Calculate hereby also $\mathrm{tr}\, T_a^{(\mathrm{adj})} T_b^{(\mathrm{adj})}$.
What is $C_{2m+1}^{(\mathrm{o})}$ for $\mathrm{SO}(N)$?
Calculate the explicit expression of C_3 for $\mathrm{SU}(3)$ in the fundamental as well as in the adjoint representation.