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Symmetries in Physics – Exercise Sheet 13

Exercise 13.1

In this exercise we explicitly construct the Cartan-Weyl bases, the Chevalley bases and the Dynkin diagrams for $\mathrm{SO}(N)$, $\mathrm{SU}(N)$, and $\mathrm{USp}(2N)$.

(i) Take the Cartan subalgebras in Exercise 12.1 and employ the basis of the three Lie algebras from Exercise 11.2. Show that the eigenvalue equation $[H_j, E_\alpha]_- = \alpha_j E_\alpha$ reduce to decoupled eigenvalue equations for 2×2 matrices for $\mathrm{SU}(N)$ and for 4×4 matrices for $\mathrm{SO}(N)$ and $\mathrm{USp}(2N)$.

(ii) Diagonalize these matrix equations and calculate the set of root vectors Δ and the eigenvectors E_α.

(iii) Calculate the angles between the root vectors and the relations between length of the root vectors. In particular, identify a set of positive root vectors, the corresponding simple root vectors α and their corresponding H_α and $E_{\pm\alpha}$. A more general definition of the set of positive root vectors Δ_+ than the one in the lecture is a set which contains one root vector of each pair of root vectors $\pm\alpha$ such that if $\alpha, \beta \in \Delta_+$ and $\alpha + \beta \in \Delta$ then $\alpha + \beta \in \Delta_+$. A simple root vector is then an element $\gamma \in \Delta_+$ such that there is no $\alpha, \beta \in \Delta_+$ with $\gamma = \alpha + \beta$.

(iv) Calculate the Cartan matrix for all three groups and construct the Dynkin-diagrams.

Exercise 13.2

We consider the group $\mathrm{SU}(3)$. This group serves as the flavor symmetry when QCD is composed of three quarks (u,d,s), only. This model is a good approximation in certain energy regimes. Some irreducible representations of this flavor group can explain the spectra. Construct all states of the following irreducible representations given by the highest weight states, $(\mu_1, \mu_2)_W$:

(i) $(1, 1)_W$ (the adjoint representation corresponding to a subset of mesons (quark-antiquark) and can be found for the baryons (three quarks) as well),

(ii) $(3, 0)_W$ (the representation corresponding to a subset of baryons). To which particles does the representation $(0, 3)_W$ correspond?

What is the dimension of the three representations? What are the eigenvalues of the states with respect to operators in the Cartan subalgebra?