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Symmetries in Physics – Exercise Sheet 14

Exercise 14.1

(i) Show that the Young tableaux dimension formula gives $(\mu_1+1)(\mu_2+1)(\mu_1+\mu_2+2)/2$ for $\mathrm{SU}(3)$, where μ_1 is the difference in length between the first and second row and μ_2 is the length of the second row.

(ii) Use the Young tableaux method to show that $3\otimes\bar{3}=8\oplus 1$. Consider the tensor $\psi_a\phi_{bc}=-\psi_a\phi_{cb}$, which is a representation of $3\otimes\bar{3}$ and can be interpreted as a quark and an anti-quark. Construct the two tensors corresponding to the two irreducible representations 8 and 1 which can be identified with the meson resonances measured in experiments. What are the transformation properties of these two tensors under the transformation $\psi_a\to U_a^b\psi_b$ and $\phi_{bc}\to U_b^{b'}U_c^{c'}\phi_{b'c'}$ with $U\in\mathrm{SU}(3)$?

(iii) Use the Young tableaux method to show that $3\otimes 3\otimes 3=10\oplus 8\oplus 8\oplus 1$. Consider the tensor $\psi_a^{(1)}\psi_b^{(2)}\psi_c^{(3)}$ which represents three quarks. Construct the four tensors corresponding to the irreducible representations 10, 8, 8, and 1. What are the transformation properties of these four tensors under the transformation $\psi_a^{(j)}\to U_a^b\psi_b^{(j)}$ with $U\in\mathrm{SU}(3)$?

Exercise 14.2

There is a more efficient way to decompose a tensor product of irreducible representations into a direct sum of irreducible representations. This technique is a result of random matrix theory and uses the fact that the characters of irreducible representations are orthogonal to each other with respect to the group invariant scalar product. Thus we have to replace the orthogonality relation for finite groups,

$$\frac{1}{|G|}\sum_{g\in G}\chi_\mu(g)\chi_\nu(g^{-1})=\delta_{\mu\nu}, \tag{1}$$

by the one for compact Lie-groups,

$$\int_G \chi_\mu(U)\chi_\nu(U^{-1})d\mu(U)=\delta_{\mu\nu}. \tag{2}$$

The normalized measure $d\mu(U)$ is known as the Haar measure of the group G and fulfills the symmetry $d\mu(U)=d\mu(VU)=d\mu(UV)$ for all $V\in G$. Recall that a character of a matrix representation is the trace of the matrix. Two crucial consequences follow from this. First we can reduce the orthogonality of the character on the whole group, cf. Eq. (2) to the orthogonality on the Cartan subspace divided by the Weyl group, i.e.

$$\int_{\mathrm{Cart}_G/\mathrm{Weyl}_G}\chi_\mu(U)\chi_\nu(U^{-1})d\widehat{\mu}(U)=\chi_\mu(\mathbf{1})\delta_{\mu\nu}, \tag{3}$$

where $d\widehat{\mu}(U)$ is the normalized Haar measure induced by $d\mu(U)$.

Second the character of a tensor product of two representations, D_1 and D_2, is the product of the characters of these two representations, i.e.

$$\chi(D_1\otimes D_2)=\chi(D_1)\chi(D_2). \tag{4}$$

With this relation we decompose products of irreducible representations for the two Lie groups $\mathrm{SU}(2)$ and $\mathrm{SU}(3)$.

(i) For the Lie-group $\mathrm{SU}(2)$ we have the set

$$\mathrm{Cart}_{\mathrm{SU}(2)}/\mathrm{Weyl}_{\mathrm{SU}(2)} = \{\mathrm{diag}(e^{\imath\varphi}, e^{-\imath\varphi}) | 0 \leq \varphi \leq \pi\} \tag{5}$$

with the measure

$$d\widehat{\mu}(U) = \frac{2}{\pi} \sin^2 \varphi d\varphi. \tag{6}$$

The character of the representation with the highest weight $(\mu_1)_W = (2l)_W$ ($l \in \mathbb{N}_0/2$ is the angular quantum number) is

$$\chi_{2l}(U) = \det \begin{bmatrix} 1 & e^{\imath(2l+1)\varphi} \\ 1 & e^{-\imath(2l+1)\varphi} \end{bmatrix} / \det \begin{bmatrix} 1 & e^{\imath\varphi} \\ 1 & e^{-\imath\varphi} \end{bmatrix} = \frac{\sin(2l+1)\varphi}{\sin\varphi}. \tag{7}$$

(a) Show that the dimension of the irreducible representation denoted by l is $\chi_{2l}(\mathbb{1}) = 2l + 1$.

(b) Decompose the tensor product of the representation $(2l_1 + 1) \otimes (2l_2 + 1)$ into irreducible representations.

(c) Decompose the tensor product of fundamental representations ($l = 1/2$), namely $\bigotimes_n 2 = \overbrace{2 \otimes 2 \otimes \cdots \otimes 2}^{n-\mathrm{times}}$ into irreducible representations.

(ii) For the Lie-group $\mathrm{SU}(3)$ we have the set

$$\mathrm{Cart}_{\mathrm{SU}(3)}/\mathrm{Weyl}_{\mathrm{SU}(3)} = \{\mathrm{diag}(e^{\imath\varphi_1}, e^{\imath\varphi_2}, e^{-\imath(\varphi_1+\varphi_2)}) | -\pi \leq -\varphi_1 - \varphi_2 \leq \varphi_2 \leq \varphi_1 \leq \pi\} \tag{8}$$

with the measure

$$d\widehat{\mu}(U) = \frac{1}{4\pi^2} \left| \det \begin{bmatrix} 1 & e^{\imath\varphi_1} & e^{2\imath\varphi_1} \\ 1 & e^{\imath\varphi_2} & e^{2\imath\varphi_2} \\ 1 & e^{-\imath(\varphi_1+\varphi_2)} & e^{-2\imath(\varphi_1+\varphi_2)} \end{bmatrix} \right|^2 d\varphi_1 d\varphi_2. \tag{9}$$

The character of the representation with the highest weight $(\mu_1, \mu_2)_W$ ($\mu_1, \mu_2 \in \mathbb{N}_0$) is

$$\chi_{\mu_1,\mu_2}(U) = \det \begin{bmatrix} e^{-\imath\mu_2\varphi_1} & e^{\imath\varphi_1} & e^{\imath(\mu_1+2)\varphi_1} \\ e^{-\imath\mu_2\varphi_2} & e^{\imath\varphi_2} & e^{\imath(\mu_1+2)\varphi_2} \\ e^{\imath\mu_2(\varphi_1+\varphi_2)} & e^{-\imath(\varphi_1+\varphi_2)} & e^{-\imath(\mu_1+2)(\varphi_1+\varphi_2)} \end{bmatrix} / \det \begin{bmatrix} 1 & e^{\imath\varphi_1} & e^{2\imath\varphi_1} \\ 1 & e^{\imath\varphi_2} & e^{2\imath\varphi_2} \\ 1 & e^{-\imath(\varphi_1+\varphi_2)} & e^{-2\imath(\varphi_1+\varphi_2)} \end{bmatrix}. \tag{10}$$

(a) Show that the dimension of the irreducible representation with the highest weight $(\mu_1, \mu_2)_W$ is $\chi_{\mu_1,\mu_2}(\mathbb{1}) = (\mu_1 + 1)(\mu_2 + 1)(\mu_1 + \mu_2 + 2)/2$.

(b) Decompose the tensor product of fundamental ($(\mu_1, \mu_2)_W = (1, 0)$) and anti-fundamental ($(\mu_1, \mu_2)_W = (0, 1)$) representations, namely $(\bigotimes_n 3) \otimes (\bigotimes_{n'} \bar{3}) = \overbrace{3 \otimes \cdots \otimes 3}^{n-\mathrm{times}} \otimes \overbrace{\bar{3} \otimes \cdots \otimes \bar{3}}^{n'-\mathrm{times}}$ into irreducible representations.