One of the simplest models for the slow relaxation and aging of glasses is the trap model by Bouchaud and others, which represents a system as a point in configuration-space hopping between local energy minima. The time evolution depends on the transition rates and the network of allowed jumps between the minima. We consider the case of sparse configuration-space connectivity given by a random graph, and study the spectral properties of the resulting master operator. We develop a general approach using the cavity method that gives access to the density of states in large systems, as well as localisation properties of the eigenvectors, which are important for the dynamics. We illustrate how, for a system with sparse connectivity and finite temperature, the density of states and the average inverse participation ratio have attributes that arise from a non-trivial combination of the corresponding mean field (fully connected) and random walk (infinite temperature) limits. In particular, we find a range of eigenvalues for which the density of states is of mean-field form but localisation properties are not, and speculate that the corresponding eigenvectors may be concentrated on extensively many clusters of network sites.